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PROPERTIES OF SOME SPECTRA OF SUPERPOSITION OPERATORS

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ABSTRACT. We consider the nonlinear superposition operator F in Banach spaces of sequences l_p, generated by the function $f(s,u)$. We analyze the Rhodius spectra $\sigma_R(F)$ and the Neuberger spectra $\sigma_N(F)$ of these operators F generated by the function

$$f(s,u) = a(s) + \phi(u),$$

where $(a(s))_{s\in\mathbb{N}}$ is a sequence from l_p $(1 \le p \le \infty)$, and $\phi(u)$ is a continuous function in $\mathbb{R}$. Some connections between the property of the function $\phi(u)$ and the corresponding spectra $\sigma_R(F)$ and $\sigma_N(F)$ are given in this paper. There are also a few examples that verify proposed theorems.

1. INTRODUCTION

In this paper we consider the Rhodius and Neuberger spectra of superposition operators in Banach spaces l_p $(1 \le p \le \infty)$. The superposition operators have an important place in many mathematical problems and also there are various applications in mathematical physics, mathematical economics, mathematical biology and so on. Let $f(s,u)$ be a function defined on $\mathbb{N}\times\mathbb{R}$ with values in $\mathbb{R}$. Given a function $x = x(s)$, by applying f, we get the function $f(s,x(s))$ and this function generates an operator F

$$F(x(s) = f(s,x(s)). \tag{1.1}$$

This operator (1.1) is called the superposition operator, composition operator or Nemytskii operator. We take $x = x(s)$ a sequence from the l_p spaces of sequences $(1 \le p \le \infty)$, which are the Banach spaces equipped with the standard norm. It is known that the spectrum of a linear operator has many useful properties. For nonlinear operators F, the notion spectrum of F is a wider concept, based on the property of $\lambda I - F$ being a regular map.

For the class of all continuous operators F on a Banach space X, denoted by $\mathfrak{C}(X)$, the following definition of Rhodius spectrum has been introduced.

Definition 1.1. *For the continuous operator $F : X \longrightarrow X$ the set*

$$\rho_R(F) = \{\lambda \in \mathbb{R} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}(X)\}$$

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is called the Rhodius resolvent set and

$$\sigma_R(F) = \mathbb{R} \setminus \rho_R(F)$$

is the Rhodius spectrum.

Thus, a point $\lambda \in \mathbb{R}$ belongs to $\rho_R(F)$ if and only if $\lambda I - F$ is a homeomorphism on X. The Neuberger spectrum of nonlinear operators was proposed for $\mathfrak{C}^1(X)$, the class of continuously Fréchet differentiable operators on Banach space X.

Definition 1.2. *For an operator $F : X \to X$, which is continuously Fréchet differentiable, the Neuberger resolvent set is defined by*

$$\rho_N(F) = \left\{\lambda \in \mathbb{R} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}^1(X)\right\}$$

and the set $\sigma_N(F) = \mathbb{R} \setminus \rho_N(F)$ is called the Neuberger spectrum of F.

A point $\lambda \in \mathbb{R}$ belongs to $\rho_N(F)$ if and only if $\lambda I - F$ is a diffeomorphism on X.

Some useful properties of these spectra are:

- If F is a linear operator $\Rightarrow \sigma_R(F) = \sigma_N(F) = \sigma(F)$,
- If $F0 = 0 \Rightarrow \sigma_R(F) \subseteq \sigma_N(F)$,
- If the underlying space is complex, then $\sigma_N(F)$ is nonempty.

More information about various nonlinear spectral theories can be found in [1]. The conditions of acting, continuity and differentiability of the superposition operator in Banach spaces l_p, are given in the following three theorems from [2].

Theorem 1.1. *Let $1 \leq p, q < \infty$. Then the following properties are equivalent:*

- *the operator F acts from l_p to l_q;*
- *there are functions $a(s) \in l_q$ and constants $\delta > 0, n \in \mathbb{N}, b \geq 0$, for which*

$$|f(s,u)| \leq a(s) + b|u|^{\frac{p}{q}} \ (s \geq n, |u| < \delta);$$

- *for any $\epsilon > 0$ there exists a function $a_\epsilon \in l_q$ and constants $\delta_\epsilon, > 0, n_\epsilon \in \mathbb{N}$, $b_\epsilon \geq 0$, for which $\|a_\epsilon(s)\|_q < \epsilon$ and*

$$|f(s,u)| \leq a_\epsilon(s) + b_\epsilon |u|^{\frac{p}{q}} \ (s \geq n_\epsilon, |u| < \delta_\epsilon).$$

Theorem 1.2. *Let $1 \leq p, q < \infty$ and let the superposition operator* (1.1)*, generated by the function $f(s,u)$, acting from l_p to l_q. Then this operator is continuous if and only if each of the functions is continuous for every $s \in \mathbb{N}$.*

Theorem 1.3. *Let $1 \leq p, q < \infty$ and the operator F generated by the function $f(s,u)$ acting from l_p into l_q. The operator F is differentiable at $x_0 \in l_p$ if and only if $f'_u(s,\cdot)$ is continuous at x_0 for almost all $s \in \mathbb{N}$.*

2. MAIN RESULTS

The Rhodius and Neuberger spectra are defined for a continuous operator F, that is, $F \in \mathfrak{C}(l_p)$, so the function $\phi(u)$ has to be continuous function on $\mathbb{R}$.

Theorem 2.1. *Let the superposition operator* $F : l_p \longrightarrow l_p$ *be generated by the function* $f(s,u) = a(s) + \phi(u)$, *where* $(a(s))_s$ *is a sequence from the space* l_p $(1 \le p \le \infty)$. *If* $\phi(u)$ *is a bijective function, then* $0 \in \rho_R(F)$ *and* $0 \notin \sigma_R(F)$

Proof. For $\lambda = 0$ the operator $\lambda I - F$ becomes $-F$. If $\phi(u)$ is a bijection then the functions $f(s,u) = a(s) + \phi(u)$ and $-f(s,u) = -a(s) - \phi(u)$ are bijective for every $s \in \mathbb{N}$. It follows that the operator F, generated by f and the operator $-F$, generated by $-f$ are bijective. Since $-f$ is a bijective function, it follows that its inverse $(-f)^{-1}$ exists and it is also a bijective function. The function $-f(s,u)$ is a bijective and continuous for every s and hence $(-f)^{-1}(s,u)$ is a bijective and continuous function for every s. From the Theorem 1.2 we conclude the operator $(-F)^{-1}$ generated by $(-f)^{-1}$ is a continuous operator. So, we show that operator $-F$ is bijective and $(-F)^{-1} \in \mathfrak{C}(l_p)$. It means that $0 \in \rho_R(F)$ and $0 \notin \sigma_R(F)$. □

The Neuberger spectrum is defined for $F \in \mathfrak{C}^1(l_p)$, so the function $\phi(u)$ has to be continuously differentiable on $\mathbb{R}$. Then the derivative $f'_u(s,u) = \phi'(u)$ is a continuous function for all $s \in \mathbb{N}$ and from Theorem 1.3 it follows the operator F is continuously Fréchet differentiable.

Theorem 2.2. *Let the superposition operator* $F : l_p \longrightarrow l_p$ *be generated by the function* $f(s,u) = a(s) + \phi(u)$, *where* $(a(s))_s$ *is a sequence from the space* l_p $(1 \le p \le \infty)$. *Let* $\phi(u)$ *be a bijective and continuously differentiable function.*
a) If $\phi'(u) \neq 0, \forall u$, *then* $0 \in \rho_N(F)$ *and* $0 \notin \sigma_N(F)$.
b) If there exists u_0 *such that* $\phi'(u_0) = 0$, *then* $0 \notin \rho_N(F)$ *and* $0 \in \sigma_N(F)$.

Proof. a) The function $\phi(u)$ is a continuously differentiable function, so $\phi'(u)$ exists for every $u \in \mathbb{R}$ and it is a continuous function. Since the function ϕ is bijective, its inverse ϕ^{-1} exists and it is also bijective. If $\phi'(u) \neq 0, \forall u$, then in virtue of the inverse function theorem (see [6]), the ϕ^{-1} is a differentiable function and

$$(\phi^{-1})'(y) = \frac{1}{\phi'(u)} \tag{2.1}$$

holds, where $\phi(u) = y$ and $\phi^{-1}(y) = u$.
We have $f'_u(s,u) = (a(s) + \phi(u))'_u = \phi'(u) \neq 0$ and

$$(f^{-1})'_u = \frac{1}{f'_u(s,u)} = \frac{1}{\phi'_u(u)}. \tag{2.2}$$

This derivative $(f^{-1})'_u$ is continuous in u because $\phi'(u)$ is a continuous function and $\phi'(u) \neq 0$ for every u. That is why, according to the Theorem 1.3, $(f^{-1})'_u$ generates a continuous operator $(F^{-1})'$, i.e. $(F^{-1}) \in \mathfrak{C}^1(l_p)$. Then clearly, $((-f)^{-1})'_u$ is also a continuous function and $(-F)^{-1} \in \mathfrak{C}^1(l_p)$. In the proof of Theorem 2.1 we have already shown that $-F$ is a bijective operator and now we see that $(-F)^{-1} \in \mathfrak{C}^1(l_p)$, so we have proved that $0 \in \rho_N(F)$ and $0 \notin \sigma_N(F)$.
b) If there exists u_0 such that $\phi'(u_0) = 0$, then the function ϕ^{-1} is not differentiable at $y_0 = \phi(u_0)$ and consequently, the partial derivatives $(f^{-1})'_u$ and $(-f^{-1})'_u$ are not continuously differentiable functions. Hence, $(-F)^{-1} \notin \mathfrak{C}^1(l_p)$ and this means that $0 \notin \rho_N(F)$ and $0 \in \sigma_N(F)$. □

2.1. **Examples**

Here we give two examples of nonlinear superposition operators and their Rhodius and Neuberger spectra, which illustrate and verify these theorems (Theorem 2.1 and Theorem 2.2).

Example 2.1. *Let the operator* $F : l_p \longrightarrow l_p$ *be generated by the function*

$$f(s,u) = a(s) + u^n, \tag{2.3}$$

where $n \geq 3$ *and* n *is an odd number. This function* $\phi(u) = u^n$ *is bijective. In [4] it was found that* $\sigma_R(F) = (0,\infty)$, *so* $0 \notin \sigma_R(F)$. *So, in this example we see that the function* $\phi(u)$ *is bijective and* $0 \notin \sigma_R(F)$ *and this confirms Theorem 2.1.*
The Fréchet derivative of the operator F *generated by* (2.3)*, at* $x_0 = (x_1, x_2, \ldots)$ *along* $h = (h_1, h_2, \ldots)$ *is:*

$$F'(x_0)h = (nx_1^{n-1}h_1, nx_1^{n-1}h_2, \ldots).$$

In [3] we found $\sigma_N(F) = [0,\infty)$, *so* $0 \in \sigma_N(F)$. *Here, the function* $\phi(u) = u^n$ *is continuously differentiable (*$\phi'(u) = n \cdot u^{n-1}$*), but its inverse* $\phi^{-1}(u) = \sqrt[n]{u}$ *is not differentiable at* $u = 0$ $\left((\phi^{-1})'(u) = \frac{1}{n} \cdot \frac{1}{\sqrt[n]{u^{n-1}}}\right)$. *In this case the function* $\phi(u) = u^n$ *is bijective and* $\phi'(u) = n \cdot u^{n-1}$. *Therefore, there exists* $u_0 = 0$, *such that* $\phi'(u_0) = \phi'(0) = 0$. *In virtue of the Theorem 2.2 it follows that* $0 \in \sigma_N(F)$. *Hence, in this example Theorem 2.2 is verified.*

Example 2.2. *Let the operator* $F : l_p \longrightarrow l_p$ *be generated by the function*

$$f(s,u) = a(s) + \sqrt[n]{u}, \tag{2.4}$$

where $n \geq 3$ *and* n *is an odd number. This function* $\phi(u) = \sqrt[n]{u}$ *is bijective and the function* $f(s,u)$ *is bijective for every* $s \in \mathbb{N}$, *so the operator* F *is bijective. In [4] we found that* $\sigma_R(F) = (0,\infty)$, *so* $0 \notin \sigma_R(F)$. *Hence,in this example the function* $\phi(u) = \sqrt[n]{u}$, *(n-odd number) is bijective and* $0 \notin \sigma_R(F)$ *and it agrees with Theorem 2.1.*
The Fréchet derivative of the operator F *generated by* (2.4)*, at* $x_0 = (x_1, x_2, \ldots)$ *along* $h = (h_1, h_2, \ldots)$ *is:*

$$F'(x_0)h = \left(\frac{1}{n\sqrt[n]{x_1^{n-1}}} h_1, \frac{1}{n\sqrt[n]{x_2^{n-1}}} h_2, \ldots \right).$$

The function $-f(s,u)$ *is bijective and the operator* $-F$ *is bijective. We have*

$$-f(s,u) = -a(s) - \sqrt[n]{u}$$

and

$$(-f)^{-1}(s,u) = (-a(s) - u)^n.$$

Let us find the partial derivative of $(-f)^{-1}$ *with respect to the variable* u :

$$((-f)^{-1})'_u = n \cdot (-u - a(s))^{n-1} \cdot (-u - a(s))'_u = -n \cdot (-u - a(s))^{n-1}. \tag{2.5}$$

The function (2.5) *is continuous for all* $s \in \mathbb{N}$ *and from Theorem 1.3 it follows that the operator* $(-F)^{-1}$ *is continuously differentiable. Here we see that* $-F$ *is a bijective operator and* $(-F)^{-1} \in \mathfrak{C}(l_p)$. *Now from Definition 1.2 it follows* $0 \in \rho_N(F)$ *and* $0 \notin$

$\sigma_N(F)$. *In this case we have the function* $\phi(u) = \sqrt[n]{u}$ *which is bijective and*

$$\phi'(u) = (\sqrt[n]{u})'_u = \frac{1}{n} \cdot \frac{1}{\sqrt[n]{u^{n-1}}} \neq 0, \forall u \in \mathbb{R}. \tag{2.6}$$

Therefore, from Theorem 2.2 it follows that $0 \in \rho_N(F)$ *and* $0 \notin \sigma_N(F)$. *So, in this example we also get the validation of Theorem 2.2.*

3. CONCLUSION

In this paper we observe the class of superposition operators $F : l_p \to l_p$, generated by the function $f(s,u) = a(s) + \phi(u)$. We find out how the fact that the function $\phi(u)$ is bijective affects the Rhodius and Neuberger spectra of the operator F, generated by the function $f(s,u)$. We conclude that it affects the corresponding spectra in regards to whether $\sigma_R(F)$ and $\sigma_N(F)$ contain 0. In [5] we found that if the function $\phi(u)$ is not a bijection, then $0 \in \sigma_R(F)$ and $0 \in \sigma_N(F)$. Our further goal is to investigate how some other properties of these spectra of nonlinear superposition operators, such as closedness, boundedness etc, depend on the properties of their generating function $f(s,u)$.

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