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Vuković, Mirjana, urednik; Nurkanović, Mehmed, urednik

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CONNECTIVITY ESTIMATES IN THE HOMOLOGICAL TAYLOR TOWER FOR THE SPACE OF REDUCED EMBEDDINGS IN $\mathbb{R}^n$

FRANJO ŠARČEVIĆ

ABSTRACT. Define $\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$, the space of reduced embeddings of a smooth manifold M in $\mathbb{R}^n$, to be the homotopy fiber of the inclusion map $\mathrm{Emb}(M,\mathbb{R}^n) \to \mathrm{Imm}(M,\mathbb{R}^n)$, where $\mathrm{Imm}(M,\mathbb{R}^n)$ is the space of immersions of M in $\mathbb{R}^n$, and denote by $\underline{H}\mathbb{Z}$ the Eilenberg-MacLane spectrum. The Taylor tower for the space $\underline{H}\mathbb{Z} \wedge \overline{\mathrm{Emb}}(M,\mathbb{R}^n)$, which is the homological version of the tower for the space $\mathrm{Emb}(M,\mathbb{R}^n)$, is known to converge under certain dimensional assumptions, meaning that the connectivity of the map from $\underline{H}\mathbb{Z} \wedge \overline{\mathrm{Emb}}(M,\mathbb{R}^n)$ to its k^{th} polynomial approximation $T_k\underline{H}\mathbb{Z} \wedge \overline{\mathrm{Emb}}(M,\mathbb{R}^n)$ approaches ∞ as k approaches ∞. Here we give a brief exposition of the known results and derive a slightly better connectivity estimate using a recent result obtained for the space of r-immersions.

1. INTRODUCTION

Manifold calculus of functors, or Goodwillie calculus, studies *good* (meaning *finitary* and *isotopy*) contravariant functors $F\colon \mathcal{O}(M) \to \mathcal{C}$, where $\mathcal{O}(M)$ is the category of open subsets of a smooth manifold M with inclusions as morphisms, and $\mathcal{C}$ is a suitable category (usually Top or Spectra).

The central question in the theory is that of the convergence of the *Taylor tower*

$$F(-) \to (T_\infty F(-) \to \cdots \to T_k F(-) \to \cdots \to T_0 F(-))$$

associated to the functor. Here $T_k F(-)$, k-th stage of the tower, is a k-th polynomial approximation of the functor, and $T_\infty F(-)$ is the inverse limit of the tower. There are two convergence questions: *intrinsic* convergence of the tower, which means that the connectivity of the map between two succesive stages $T_{k+1}F(-) \to T_k F(-)$ approaches ∞ as k approaches ∞, and the convergence *to* the tower, which means that there exists a weak equivalence between $F(-)$ and $T_\infty F(-)$.

Define $\mathrm{Emb}(M,\mathbb{R}^n)$ to be the space of embedding of M in $\mathbb{R}^n$. The central result of the Goodwillie calculus is the Goodwillie-Klein-Weiss theorem which, in a special case, says that the map

$$T_{k+1}\,\mathrm{Emb}(M,\mathbb{R}^n) \to T_k\,\mathrm{Emb}(M,\mathbb{R}^n)$$

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is $(k(n-m-2)-m+1)$-connected and that the map

$$\mathrm{Emb}(M,\mathbb{R}^n)\to T_k\,\mathrm{Emb}(M,\mathbb{R}^n)$$

is $(k(n-m-2)-m+1)$-connected. Therefore, as long as $n>m+2$, the Taylor tower

$$\mathrm{Emb}(M,\mathbb{R}^n)\to(T_\infty\,\mathrm{Emb}(M,\mathbb{R}^n)\to\cdots\to T_k\,\mathrm{Emb}(M,\mathbb{R}^n)\to\cdots\to T_0\,\mathrm{Emb}(M,\mathbb{R}^n))$$

converges intrinsically and to the tower. The details can be found in [2,3,5,6].

This convergence is actually homotopical convergence, because the connectivity in question here is the homotopical one. We can also consider the homological version of the Taylor tower for $\mathrm{Emb}(M,\mathbb{R}^n)$. Taking the smash product $\wedge$ of the Eilenberg-MacLane spectrum $\underline{H}\mathbb{Z}$ with a based space X produces the spectrum $\underline{H}\mathbb{Z}\wedge X$ whose homotopy is equivalent to the reduced homology of the space X; more precisely, there exists an isomorphism $\pi_i(\underline{H}\mathbb{Z}\wedge X)\cong\widetilde{\mathrm{H}}_i(X;\mathbb{Z})$.

Thus, we consider the Taylor tower for the space

$$\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

to be the *homological Taylor tower* for $\mathrm{Emb}(M,\mathbb{R}^n)$, where we have replaced the space of embeddings with embeddings modulo immersions defined by

$$\overline{\mathrm{Emb}}(M,\mathbb{R}^n)=\mathrm{hofiber}\,(\mathrm{Emb}(M,\mathbb{R}^n)\to\mathrm{Imm}(M,\mathbb{R}^n)),$$

which is convenient (to cancel the tangential data of the immersion).

As shown in [8], the connectivity of the map

$$\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\to T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\tag{1.1}$$

is

$$(k+1)\left(\frac{n}{2}-m-\frac{1}{2}\right)\tag{1.2}$$

and the tower converges for $n>2m+1$.

Actually, when we write $\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$ we really mean the *taming* of the functor $\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(-,\mathbb{R}^n)$, evaluated on a *tame* manifold M, which is the interior of a compact manifold with boundary. Namely, even if a cofunctor $F(-)\colon\mathcal{O}(M)\to\mathrm{Top}$ is good, the cofunctor $\underline{J}\wedge F(-)\colon\mathcal{O}(M)\to\mathrm{Spectra}$ for a fixed spectrum $\underline{J}$ is not good [4,8], but the taming of this functor *is* good. Therefore, when evaluated on a *tame subset* of M – an element of $\mathcal{O}(M)$ which is the interior of a compact smooth codimension zero submanifold od M – the taming of a functor is equivalent to the functor. So, when evaluated on tame manifolds, there is no difference between $\underline{J}\wedge F(M)$ and the taming of it.

Here we will provide a stronger connectivity estimate for the map (1.1) (Proposition 2.1). It is

$$(k+1)\left(\frac{n}{2}-m-\frac{1}{2}\right)+(n-1)\left(\frac{1}{2}+\frac{k\bmod 2}{2}\right).$$

Prior to that, let us present two of the three results on which this story is based. The notion of analyticity is explained in the cited literature.

Theorem 1.1 ([5]). *Let F be a ρ-analytic good cofunctor with excess c and U the interior of a smooth compact codimension 0 submanifold of M of handle index $q < \rho$. Then $F(U) \to T_k F(U)$ is $(c + (k+1)(\rho - q))$-connected.*

Weiss provided the following result.

Theorem 1.2 ([8]). *If a good cofunctor $F\colon \mathcal{O}(M) \to$ Top is ρ-analytic with excess $c < 0$, where $\rho + \frac{c}{l} > m$, such that $T_{l-1}F(-)$ vanishes for some $l > 0$, and $\underline{J}$ is a (-1)-connected CW-spectrum, then the taming of the functor $\underline{J} \wedge F(-)$ is $\left(\rho + \frac{c}{l}\right)$-analytic with excess 0.*

2. ESTIMATES

It is known that the functor $F(-) = \overline{\mathrm{Emb}}(-, \mathbb{R}^n)$ is $(n-2)$-analytic with excess $3-n$ [2,3,5].

The spectrum $\underline{H}\mathbb{Z}$ is a (-1)-connected CW-spectrum, because it does not have nontrivial homotopy groups in negative dimensions (actually, it is nontrivial $\mathbb{Z}$ only in the 0-th dimension).

Also, $T_1 F(-)$ vanishes because $T_1\,\mathrm{Emb}(-, \mathbb{R}^n) \simeq \mathrm{Imm}(-, \mathbb{R}^n)$ and $T_1\,\mathrm{Imm}(-, \mathbb{R}^n) \simeq \mathrm{Imm}(-, \mathbb{R}^n)$ [7], so $l = 2$ in terms of Theorem 1.2.

It follows from Theorem 1.2 that the functor $\underline{H}\mathbb{Z} \wedge \overline{\mathrm{Emb}}(-, \mathbb{R}^n)$ is $\left(\frac{n}{2} - \frac{1}{2}\right)$-analytic with excess 0.

Now from Theorem 1.1 and the remarks on tameness it follows that the map

$$\underline{H}\mathbb{Z} \wedge \overline{\mathrm{Emb}}(M, \mathbb{R}^n) \to T_k \underline{H}\mathbb{Z} \wedge \overline{\mathrm{Emb}}(M, \mathbb{R}^n)$$

is

$$(k+1)\left(\frac{n}{2} - m - \frac{1}{2}\right)\text{-connected.}$$

In [1] the authors study the space of r-immersions, which are the immersions without r-fold self intersections. That is, the space $\mathrm{rImm}(M, \mathbb{R}^n)$ of r-immersions of M in $\mathbb{R}^n$ is the space of immersions of M in $\mathbb{R}^n$ with the property that at most $r-1$ points of M are mapped to the same point in $\mathbb{R}^n$.

The part of the central result is the following:

Theorem 2.1 ([1]). *When $r \leq n+1$, the map*

$$T_k \underline{H}\mathbb{Z} \wedge \overline{\mathrm{rImm}}(M, \mathbb{R}^n) \to T_{k-1} \underline{H}\mathbb{Z} \wedge \overline{\mathrm{rImm}}(M, \mathbb{R}^n)$$

is

$$k\left(n\frac{r-1}{r} - m - \frac{1}{r}\right) - \frac{k \bmod r}{r}(r - n - 1)\text{-connected.}$$

The tower converges intrinsically if

$$n > \frac{rm+1}{r-1}.$$

As is clear from the definition, injective immersions are just 2-immersions. If M is compact, then injective immersions are the same thing as embeddings. That is, for M compact,

$$\mathrm{Emb}(M, \mathbb{R}^n) = 2\,\mathrm{Imm}(M, \mathbb{R}^n).$$

The same is true in a more general case relevant to us: when M is tame, the space $2\,\mathrm{Imm}(M,\mathbb{R}^n)$ is equivalent to the space $\mathrm{Emb}(M,\mathbb{R}^n)$.

So, in our consideration, letting $r = 2$ in Theorem 2.1 we get a result for the space of reduced embeddings.

Corollary 2.1. *The connectivity of the map*

$$T_{k+1}\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\longrightarrow T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

is

$$(k+1)\left(\frac{n}{2}-m-\frac{1}{2}\right)+(n-1)\left(\frac{1}{2}+\frac{k \bmod 2}{2}\right).$$

The tower converges intrinsically if

$$n > 2m+1.$$

Proposition 2.1. *Let M be a tame manifold. The connectivity of the map*

$$\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\longrightarrow T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

is

$$(k+1)\left(\frac{n}{2}-m-\frac{1}{2}\right)+(n-1)\left(\frac{1}{2}+\frac{k \bmod 2}{2}\right).$$

Proof. It is known and easy to prove that, if $f\colon X\longrightarrow Y$ and $g\colon Y\longrightarrow Z$ are k-connected maps, then $g\circ f\colon X\longrightarrow Z$ is also a k-connected map. If a map is k-connected, then it is j-connected for all $j\le k$. Now, if f is ∞-connected (i.e. a weak equivalence), then f is also k-connected for all k, so g and $g\circ f$ have the same connectivity.

Using the fact that the map

$$\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\longrightarrow T_\infty\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

is a weak equivalence, this means that the connectivities c_1 and c_2 in the diagram

$$\begin{tikzcd}
\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n) \arrow[r, "\sim"] \arrow[dr, "c_2"] & T_\infty\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n) \arrow[d, "c_1"] \\
 & T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)
\end{tikzcd}$$

are the same.

Also, if the map

$$T_{k+1}\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\longrightarrow T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

is c-connected, then the connectivity of the map

$$T_\infty\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\longrightarrow T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

is at least c.

This, together with Corollary 2.1 implies that the map

$$T_\infty\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\longrightarrow T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

is also

$$(k+1)\left(\frac{n}{2}-m-\frac{1}{2}\right)+(n-1)\left(\frac{1}{2}+\frac{k \bmod 2}{2}\right)\text{-connected.}$$

That finally implies that the map

$$\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)\to T_k\underline{H}\mathbb{Z}\wedge\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$$

is

$$(k+1)\left(\frac{n}{2}-m-\frac{1}{2}\right)+(n-1)\left(\frac{1}{2}+\frac{k \bmod 2}{2}\right)\text{-connected.}$$

□

The connectivity estimate (1.2) is improved by the number

$$(n-1)\left(\frac{1}{2}+\frac{k \bmod 2}{2}\right).$$

If k is odd, this number is $n-1$; if k is even, this number is $\frac{1}{2}(n-1)$.

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Franjo Šarčević
University of Sarajevo
Department of Mathematics and Computer Science
Zmaja od Bosne 33-35, 71000 Sarajevo
e-mail: *franjo.sarcevic@pmf.unsa.ba*
URL: *pmf.unsa.ba/franjos*