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SOLVING FIRST-ORDER AND SECOND-ORDER DIFFERENCE EQUATIONS USING LIE SYMMETRIES

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Dedicated to the 75th birthday of our dear Professor Mirjana Vuković

ABSTRACT. There are well-developed algorithms for solving certain nonlinear difference equations with constant coefficients. On the other hand, nonlinear difference equations, especially with variable coefficients, are very complex. Namely, there are no universal methods of solving them. Nevertheless, difference equation methods, especially Lie symmetry groups, have been successfully used for certain classes of these equations. Using Lie symmetries, it is possible to construct the characteristics of a given equation. Then, with the help of canonical coordinates, it is possible to successfully solve some linear and non-linear difference equations of the first and second order with variable coefficients. The method of reducing the order of nonlinear difference equations can also, with certain specificities, be used successfully when solving difference equations.

The mentioned methods are illustrated in several characteristic examples.

1. INTRODUCTION AND PRELIMINARIES

It is well known that Lie symmetries can sometimes be successfully used to solve differential equations. In this paper, we will show that this is also possible in the case of difference equations by analogy with differential equations. For this reason, we will first familiarize ourselves with the essential characteristics of Lie symmetries, [1, 2].

The symmetry of a geometric object is an invertible transformation that maps the object into itself. The set of all symmetries $\mathcal{G}$ of a geometric object is a group. Symmetries $\Gamma_1, ..., \Gamma_k$ are generators of the group $\mathcal{G}$ if each symmetry can be written as a product of some symmetries Γ_i and their inversions. A differential or difference equation transformation is a symmetry if every solution of the transformed equation is also a solution of the initial equation and vice versa.

We will only consider translations, reflections, and rotations (each of which is rigid). Let us look at scaling transformations

$$\Gamma_\varepsilon : u_n \longmapsto \widehat{u_n} = e^{\varepsilon} u_n \tag{1.1}$$

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to a scalar linear homogeneous difference equation of order p. If $U_1(n), \ldots, U_p(n)$ are linearly independent solutions, then the general solution is given by

$$u_n = \sum_{i=1}^{p} c_i U_i(n).$$

Scaling (1) maps this solution into

$$\widehat{u_n} = \sum_{i=1}^{p} \widehat{c_i} U_i(n) \quad (\widehat{c_i} = e^{\varepsilon} c_i),$$

so that the set of all solutions is mapped (invertible) into itself; thus, Γ_ε is a symmetry of a difference equation for each $\varepsilon \in \mathbb{R}$.

Here, $\widehat{u_n}$ is a smooth function of u_n. Really, $\Gamma : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, a smooth invertible map whose inverse is also smooth. The set of transformations $G = \{\Gamma_\varepsilon : \varepsilon \in \mathbb{R}\}$ is a group with a composition $\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$, for all $\delta, \varepsilon \in \mathbb{R}$. Here, Γ_0 is the identical map, and $\Gamma_\varepsilon^{-1} = \Gamma_{-\varepsilon}$ holds. In addition, $\widehat{u_n}$ is an analytic function of the parameter ε. An important feature of this group is: Γ_ε is close identity transformation for every small enough ε. Suppose these close identities of the symmetry transformation are given by different equations of the pth order. In that case, the individual solution will be mapped into a one-parameter family of close solutions whose arbitrary constants depend analytically on ε. This property can be used to solve various first-order equations, which need not be linear, as will be demonstrated later.

Definition 1.1. *A parameterized set of transformations by points*

$$\Gamma_\varepsilon : X \longmapsto \widehat{X}(X;\varepsilon), \ \ \varepsilon \in (\varepsilon_0, \varepsilon_1), \ \ \varepsilon_0 < 0, \ \varepsilon_1 > 0,$$

is called a one-parameter local Lie group if the following conditions apply:

1. Γ_0 *is an identical map, so* $\widehat{X} = X$, *for* $\varepsilon = 0$,

2. $\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$ *for every* δ, ε *close enough to zero,*

3. *Each* $\widehat{x}^\alpha$ *can be represented as a Taylor series in* ε *(about* $\varepsilon = 0$ *which is determined by* X*)*

$$\widehat{x}^\alpha(X;\varepsilon) = x^\alpha + \varepsilon \xi^\alpha(X) + O\left(\varepsilon^2\right), \quad \alpha = 1, \ldots, N.$$

It follows from conditions **1.** and **2.** that $\Gamma_\varepsilon^{-1} = \Gamma_{-\varepsilon}$ when $|\varepsilon|$ is small enough. Despite its name, a local Lie group does not need to be a group; it is only necessary to satisfy the axioms of the group for small enough parameter values.
In general, the local one-parameter Lie group of symmetries of a given scalar difference equation will depend on both n and the variable u_n [1].

Example 1.1. *The general solution of the difference equation*

$$u_{n+1} = \frac{n+1}{n} u_n, \quad n \geq 1, \tag{1.2}$$

is $u_n = c_1 n$. *Any transformation of the form*

$$(\widehat{n}, \widehat{u_n}) = (n, u_n + \varepsilon n) \tag{1.3}$$

is the symmetry that maps $u_n = c_1 n$ *into* $\widehat{u_n} = (c_1 + \varepsilon) n$. *For each* $n \geq 1$, *(1.3) defines a one-parameter local Lie group of translations.*

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2. Characteristics and canonical coordinates

The following considerations will be limited to Lie symmetries for which $\widehat{u_n}$ depends only on n and u_n. These are the so-called Lie point symmetries that are of the form

$$\widehat{n} = n, \quad \widehat{u_n} = u_n + \varepsilon \mathcal{K}(n, u_n) + O\left(\varepsilon^2\right). \tag{2.1}$$

To see how such symmetries transform the shifted variable u_{n+k}, we simply replace n with $n+k$ in (2.2):

$$\widehat{u_{n+k}} = u_{n+k} + \varepsilon \mathcal{K}(n+k, u_{n+k}) + O\left(\varepsilon^2\right). \tag{2.2}$$

The formula (2.2) represents the *formula prolongations* for pointwise Lie symmetries.

The function $\mathcal{K}(n, u_n)$ is the *characteristic* of the local Lie group with respect to the coordinates (n, u_n). For example, the characteristic of vertical translation $(\widehat{n}, \widehat{u_n}) = (n, u_n + \varepsilon)$ is of the form

$$\mathcal{K}(n, u_n) = 1. \tag{2.3}$$

Let us observe the effect of changing coordinates from (n, u_n) in (n, v_n), where $v'(n, u_n) \neq 0$ $\left(v' = \dfrac{\partial v}{\partial u_n}\right)$. When (2.2) is a symmetry for every ε close enough to zero, we can apply Taylor's theorem to get

$$\begin{aligned} \widehat{v_n} &= v(n, \widehat{u_n}) = v\left(n, u_n + \varepsilon \mathcal{K}(n, u_n) + O\left(\varepsilon^2\right)\right) \\ &= v_n + \varepsilon v'(n, u_n) \mathcal{K}(n, u_n) + O\left(\varepsilon^2\right) \end{aligned} \tag{2.4}$$

Therefore, the characteristic with respect to (n, v_n) is equal to $\widetilde{\mathcal{K}}(n, v_n)$, where

$$\widetilde{\mathcal{K}}(n, v(n, u_n)) = v'(n, u_n) \mathcal{K}(n, u_n). \tag{2.5}$$

The values of $\widetilde{\mathcal{K}}$ and $\mathcal{K}$ will differ at most points (n, u_n), where $v'(n, u_n) \neq 1$, including only points where $\mathcal{K}(n, u_n) = 0$. When $\mathcal{K}(n, u_n) \neq 0$, then it is especially useful to introduce the *canonical coordinate* s_n, so that the translation symmetries of s_n are:

$$(\widehat{n}, \widehat{s}) = (n, s_n + \varepsilon), \quad \varepsilon \in \mathbb{R}. \tag{2.6}$$

The characteristic in relation to (n, s_n) is $\widetilde{\mathcal{K}}(n, s_n) = 1$, so due to (2.5)

$$s(n, u_n) = \int \frac{du_n}{\mathcal{K}(n, u_n)}. \tag{2.7}$$

For each n, the possible values of u_n lie on the real line, which is (typically) divided into intervals where we have omitted each value u_n for which $\mathcal{K}(n, u_n) = 0$. The equality (2.7) defines the canonical coordinate s (locally) on each interval, but it is to be expected that different coordinates correspond to different intervals. For example, if $\mathcal{K}(n, u_n) = u_n^2 - 1$, the appropriate real-valued canonical coordinate depends on u_n, as follows:

$$s(n, u_n) = \int \frac{du_n}{u_n^2 - 1} = \begin{cases} \frac{1}{2} \ln \frac{u_n - 1}{u_n + 1}, & |u_n| > 1 \\ \frac{1}{2} \ln \frac{1 - u_n}{1 + u_n}, & |u_n| < 1. \end{cases} \tag{2.8}$$

In this case $s\left(n,\frac{1}{u_n}\right)=s(n,u_n)$ for every non-zero u_n so that the map from u_n in s is not injective, which cannot be invertible unless it is predetermined that $|u_n|\geq 1$. The most significant benefit of canonical coordinates is that they simplify or even solve the given difference equation. The idea is to write the difference equation in a simpler form for s; if a more straightforward difference equation can be solved, all that remains is to write the solution over the original variables. To use this approach, one must be able to invert the map from u_n to s (at least for all points (n,u_n) that occur in any solution of the original differential equation and satisfy $\mathcal{K}(n,u_n)\neq 0$). Any coordinate s that meets this requirement will be called *compatible* with the given difference equation [1,2,7].

For any compatible canonical coordinate we can replace n with $n+k$, to obtain

$$s_{n+k}=s(n+k,u_{n+k})=E^k s,\quad k\in\mathbb{Z}. \tag{2.9}$$

According to the prolongation formula, Lie symmetry $\widehat{s}=s+\varepsilon$ prolongs into

$$\widehat{s_{n+k}}=s_{n+k}+\varepsilon. \tag{2.10}$$

3. SOLVING FIRST-ORDER DIFFERENCE EQUATIONS USING LIE SYMMETRIES

Consider the following first-order difference equation

$$u_{n+1}=w(n,u_n). \tag{3.1}$$

To map the set of solutions of (3.1) into itself, the following symmetry condition must be satisfied

$$\widehat{u_{n+1}}=w(\widehat{n},\widehat{u_n})\quad\text{when}\quad u_{n+1}=w(n,u_n). \tag{3.2}$$

Example 3.1. *([1], Problem 2.2) Find the characteristic $\mathcal{K}(n,u_n)$ for the difference equation*

$$u_{n+1}=\frac{nu_n-1}{u_n+n}, \tag{3.3}$$

and then solve this equation.

Solution. Equation (3.1) is the well-known Riccati equation [4–6]. Starting from the formula

$$\mathcal{K}(n+1,u_{n+1})=w'(n,u_n)\mathcal{K}(n,u_n)$$

where

$$w'(n,u_n)=\frac{d}{du_n}\left(\frac{nu_n-1}{u_n+n}\right)=\frac{nu_n+n^2-nu_n+1}{(u_n+n)^2}=\frac{n^2+1}{(u_n+n)^2},$$

then we have

$$\mathcal{K}(n+1,u_{n+1})=\frac{n^2+1}{(u_n+n)^2}\mathcal{K}(n,u_n). \tag{3.4}$$

We look for the characteristic $\mathcal{K}(n,u_n)$ in the form

$$\mathcal{K}(n,u_n)=\alpha_n u_n^2+\beta_n u_n+\gamma_n,$$

where $\alpha_n,\beta_n,\gamma_n$ are the coefficients which should be determined from (3.4). Thus, we have

$$\alpha_{n+1}u_{n+1}^2+\beta_{n+1}u_{n+1}+\gamma_{n+1}=\frac{n^2+1}{(u_n+n)^2}\left(\alpha_n u_n^2+\beta_n u_n+\gamma_n\right),$$

that is

$$\alpha_{n+1}\left(\frac{nu_n-1}{u_n+n}\right)^2+\beta_{n+1}\frac{nu_n-1}{u_n+n}+\gamma_{n+1}=\frac{n^2+1}{(u_n+n)^2}\left(\alpha_n u_n^2+\beta_n u_n+\gamma_n\right),$$

or

$$\alpha_{n+1}(nu_n-1)^2+\beta_{n+1}(nu_n-1)(u_n+n)+\gamma_{n+1}(u_n+n)^2=\left(n^2+1\right)\left(\alpha_n u_n^2+\beta_n u_n+\gamma_n\right).$$

By equalizing the coefficients that are found with u_n^2, u_n, and free members, we get a system

$$\begin{aligned} n^2\alpha_{n+1}+n\beta_{n+1}+\gamma_{n+1} &= \left(n^2+1\right)\alpha_n \\ -2n\alpha_{n+1}+\left(n^2-1\right)\beta_{n+1}+2n\gamma_{n+1} &= \left(n^2+1\right)\beta_n \\ \alpha_{n+1}-n\beta_{n+1}+n^2\gamma_{n+1} &= \left(n^2+1\right)\gamma_n. \end{aligned}$$

By adding the first and third equations of the last system, we have

$$\alpha_{n+1}+\gamma_{n+1}=\alpha_n+\gamma_n=c_1. \tag{3.5}$$

Subtracting from the first equation of the system the second equation multiplied by n, and then subtracting the third, also multiplied by n, we obtain

$$n\left(n^2+1\right)\alpha_{n+1}+\left(n^2+1\right)\beta_{n+1}-n\left(n^2+1\right)\gamma_{n+1}=\left(n^2+1\right)(n\alpha_n-\beta_n-n\gamma_n),$$

i.e.,

$$n(\alpha_{n+1}-\gamma_{n+1})-n(\alpha_n-\gamma_n)=-(\beta_{n+1}+\beta_n). \tag{3.6}$$

By replacing (3.5) in (3.6), we get

$$2n(\alpha_{n+1}-\alpha_n)=-(\beta_{n+1}+\beta_n). \tag{3.7}$$

If we include (3.5) in the first equation of the system, we have

$$n^2(\alpha_{n+1}-\alpha_n)+n\beta_{n+1}+c_1-\alpha_{n+1}=\alpha_n,$$

and now, considering (3.7),

$$-n^2\frac{1}{2n}(\beta_{n+1}+\beta_n)+n\beta_{n+1}+c_1-\alpha_{n+1}=\alpha_n,$$

from which

$$\beta_{n+1}-\beta_n=\frac{2}{n}(\alpha_{n+1}-\alpha_n-c_1). \tag{3.8}$$

Adding (3.7) and (3.8) gives

$$-\frac{n^2+1}{n}\alpha_{n+1}+\frac{n^2-1}{n}\alpha_n+\frac{1}{n}c_1=\beta_n. \tag{3.9}$$

Finally, by substituting (3.9) into the first equation of the system, we obtain

$$n^2\alpha_{n+1}+n\left(-\frac{(n+1)^2+1}{n+1}\alpha_{n+2}+\frac{(n+1)^2-1}{n+1}\alpha_{n+1}+\frac{1}{n}c_1\right)+$$
$$+c_1-\alpha_{n+1}=\left(n^2+1\right)\alpha_n.$$

After arranging the last expression, we get the following inhomogeneous linear difference equation

$$n\left(n^2+2n+2\right)\alpha_{n+2}-\left(2n^3+3n^2-n-1\right)\alpha_{n+1}+(n+1)\left(n^2+1\right)\alpha_n=(2n+1)\,c_1.$$

One particular solution to this equation is $\alpha_n=\frac{c_1}{2}$. Namely, if we assume that $\alpha_n=(An+B)\,c_1$, the particular solution will be obtained. It implies that $\gamma_n=\frac{c_1}{2}$, and that $\beta_n=0$ must hold (considering all three equations of the given system). Therefore, the characteristic is of the form

$$\mathcal{K}(n,u_n)=\frac{c_1}{2}\left(u_n^2+1\right).$$

If $c_1=2$, then we have

$$s_n=\int\frac{1}{\mathcal{K}(n,u_n)}du_n=\int\frac{1}{u_n^2+1}du_n=\arctan u_n. \tag{3.10}$$

Thus,

$$\begin{aligned}s_{n+1}-s_n&=\arctan u_{n+1}-\arctan u_n=\arctan\frac{u_{n+1}-u_n}{1+u_{n+1}u_n}\\&=\arctan\frac{\frac{nu_n-1}{u_n+n}-u_n}{1+\frac{nu_n^2-u_n}{u_n+n}}=\arctan\left(-\frac{1}{n}\right)=-\arctan\frac{1}{n}.\end{aligned}$$

from which

$$s_n=s_1-\sum_{i=1}^{n-1}\arctan\frac{1}{i}. \tag{3.11}$$

Using the property for the sum of the function arctan, we have

$$\sum_{i=1}^{n-1}\arctan\frac{1}{i}=\arctan A(n).$$

From (3.11), due to (3.10), it follows that

$$\arctan u_n=\arctan u_1-\arctan A(n)=\arctan\frac{u_1-A(n)}{1+u_1A(n)},$$

that is

$$u_n=\frac{u_1-A(n)}{1+u_1A(n)}\quad(n=2,3,\ldots). \tag{3.12}$$

Let us check for the first few members using the iteration procedure and then the formula (3.12).

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$$\arctan\frac{1}{1} = \arctan A(1) \Longrightarrow A(1) = 1$$

$$\arctan\frac{1}{1} + \arctan\frac{1}{2} = \arctan A(2) \Longrightarrow \arctan A(2) = \arctan\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \arctan 3$$

$$\Longrightarrow A(2) = 3$$

$$\arctan\frac{1}{1} + \arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan A(3) \Longrightarrow A(3) = \frac{3+\frac{1}{3}}{1-3\cdot\frac{1}{3}} = \infty.$$

a) Iterative

$$n = 1 \Longrightarrow u_2 = \frac{u_1 - 1}{u_1 + 1}$$

$$n = 2 \Longrightarrow u_3 = \frac{2u_2 - 1}{u_2 + 2}\,\frac{\frac{2u_1-2}{u_1+1} - 1}{\frac{u_1-1}{u_1+1} + 2} = \frac{u_1 - 3}{3u_1 + 1}$$

$$n = 3 \Longrightarrow u_4 = \frac{3u_3 - 1}{u_3 + 3}\,\frac{\frac{3u_1-9}{3u_1+1} - 1}{\frac{u_1-3}{3u_1+1} + 3} = -\frac{1}{u_1}.$$

b) According to the formula (3.12), it follows that

$$n = 1 \Longrightarrow u_2 = \frac{u_1 - 1}{u_1 + 1}$$

$$n = 2 \Longrightarrow u_3 = \frac{u_1 - A(2)}{1 + u_1 A(2)} = \frac{u_1 - 3}{1 + 3u_1}$$

$$n = 3 \Longrightarrow u_4 = \frac{u_1 - A(3)}{1 + u_1 A(3)} = \frac{\frac{u_1}{A(3)} - 1}{\frac{1}{A(3)} + u_1} = -\frac{1}{u_1}.$$

So, the formula (3.12) really gives the solution of the considered equation.

4. Solving second-order difference equations using Lie symmetries

Consider the following second-order difference equation

$$u_{n+2} = w(n, u_{n+1}, u_n). \tag{4.1}$$

The so-called LSC condition for the difference equation (4.1) is of the form

$$\mathcal{K}(n+2, w) - D_2 w \mathcal{K}(n+1, u_{n+1}) - D_1 w \mathcal{K}(n, u_n) = 0. \tag{4.2}$$

The first term in LSC (4.2), with assumptions $D_1 w \neq 0$, $D_2 w \neq 0$, is eliminated by applying the following differential operator

$$\left(\frac{1}{D_1 w}\right)\frac{\partial}{\partial u_n} - \left(\frac{1}{D_2 w}\right)\frac{\partial}{\partial u_{n+1}}, \tag{4.3}$$

which gives

$$\mathcal{K}'(n+1, u_{n+1}) + D_2\eta\mathcal{K}(n+1, u_{n+1}) - \mathcal{K}'(n, u_n) + D_1\eta\mathcal{K}(n, u_n) = 0, \tag{4.4}$$

where $\eta\left(n,u_n,u_{n+1}\right)=\ln\left|\dfrac{D_2w}{D_1w}\right|$. Then $\mathcal{K}'\left(n+1,u_{n+1}\right)$ can be eliminated by differentiating (4.4) with respect to u_n, which gives

$$D_{12}\eta\mathcal{K}\left(n+1,u_{n+1}\right)-\mathcal{K}''\left(n,u_n\right)+D_1\eta\mathcal{K}'\left(n,u_n\right)+D_{11}\eta\mathcal{K}\left(n,u_n\right)=0. \tag{4.5}$$

At this stage, we have two possibilities.

First, if $D_{12}\eta=0$, then (4.5) can be integrated to obtain

$$\mathcal{K}'\left(n,u_n\right)-D_1\eta\mathcal{K}\left(n,u_n\right)=\alpha\left(n\right). \tag{4.6}$$

Now, replacing (4.6) with (4.4) (to eliminate $\mathcal{K}'\left(n,u_n\right)$ and $\mathcal{K}'\left(n+1,u_{n+1}\right)$) we get

$$\left(ED_1\eta+D_2\eta\right)\mathcal{K}\left(n+1,u_{n+1}\right)=\alpha\left(n\right)-\alpha\left(n+1\right). \tag{4.7}$$

If $D_2\eta=-ED_1\eta$, then from (4.7) it follows that $\alpha\left(n\right)=c_1$ and

$$\mathcal{K}\left(n,u_n\right)=\frac{\alpha\left(n-1\right)-\alpha\left(n\right)}{D_1\eta+E^{-1}D_2\eta}, \tag{4.8}$$

which can be replaced in (4.6), thus obtaining (at most) a difference equation of the first order in $\alpha\left(n\right)$. In the second case, the function $\mathcal{K}\left(n,u_n\right)$ which results from (4.4) must be replaced by (4.2), and any additional constraints this creates must be resolved.

Another possibility is to be $D_{12}\eta\neq 0$, when the equation (4.5) needs to be divided by $D_{12}\eta$ and then differentiated once more by u_n. The coefficients of the resulting difference equation may depend on u_{n+1}. If this happens, then the equation should be separated into a system of equations whose coefficients depend only on n and u_n. Then, continue the solving process as before.

Example 4.1. *([1], Ex. 2.17) Determine all characteristics of Lie point symmetries for the difference equation*

$$u_{n+2}=\frac{1}{u_{n+1}+u_n}-u_{n+1}-2\left(-1\right)^n\quad\left(n\geq 0\right). \tag{4.9}$$

Solution. Here, we give a much more detailed solution than in [1].
The LSC for a given difference equation is of the form

$$\mathcal{K}\left(n+2,\frac{1}{u_{n+1}+u_n}-u_{n+1}-2\left(-1\right)^n\right)+\left(1+\frac{1}{\left(u_{n+1}+u_n\right)^2}\right)\mathcal{K}\left(n+1,u_{n+1}\right)+$$
$$+\frac{1}{\left(u_{n+1}+u_n\right)^2}\mathcal{K}\left(n,u_n\right)=0, \tag{4.10}$$

because $D_1w=-\dfrac{1}{\left(u_{n+1}+u_n\right)^2}$, $D_2w=-\dfrac{1}{\left(u_{n+1}+u_n\right)^2}-1$, so

$$\eta=\ln\left|\frac{D_2w}{D_1w}\right|=\ln\left(\left(u_{n+1}+u_n\right)^2\right)+1.$$

Since $D_{12}\eta\neq 0$, then the equation

$$D_{12}\eta\mathcal{K}\left(n+1,u_{n+1}\right)-\mathcal{K}''\left(n,u_n\right)+D_1\eta\mathcal{K}'\left(n,u_n\right)+D_{11}\eta\mathcal{K}\left(n,u_n\right)=0, \tag{4.11}$$

should be divided by $D_{12}\eta = \dfrac{2\left(1-(u_{n+1}+u_n)^2\right)}{\left((u_{n+1}+u_n)^2+1\right)^2}$, and we get (due to $D_{11}\eta = D_{12}\eta$)

$$\mathcal{K}(n+1,u_{n+1}) + \frac{\left((u_{n+1}+u_n)^2+1\right)^2}{2\left((u_{n+1}+u_n)^2-1\right)}\mathcal{K}''(n,u_n) -$$
$$-\frac{(u_{n+1}+u_n)\left((u_{n+1}+u_n)^2+1\right)}{(u_{n+1}+u_n)-1}\mathcal{K}'(n,u_n) + \mathcal{K}(n,u_n) = 0. \tag{4.12}$$

After differentiation by u_n, $\mathcal{K}(n+1,u_{n+1})$ is lost, and we get

$$\left((u_{n+1}+u_n)^4-1\right)\mathcal{K}'''(n,u_n) - 4(u_{n+1}+u_n)\mathcal{K}''(n,u_n) + 4\mathcal{K}'(n,u_n). \tag{4.13}$$

All coefficients in the equation (4.13) depend on u_{n+1} so the equation can be decomposed into a system of differential equations, each of which can be multiplied by a special power of $(u_{n+1}+u_n)$

$$\mathcal{K}'''(n,u_n) = 0, \;\; \mathcal{K}''(n,u_n) = 0, \;\; \mathcal{K}'(n,u_n) = 0.$$

From here, $\mathcal{K}(n,u_n) = \alpha(n)$, so by replacing it in (4.12), we obtain

$$\alpha(n+1) + \alpha(n) = 0 \Longrightarrow \alpha(n) = c_1(-1)^n.$$

Now, we can express the characteristic $\mathcal{K}(n,u_n) = \alpha(n) = c_1(-1)^n$, and substituting into (4.10) we have

$$c_1(-1)^{n+2} + \left(1+\frac{1}{(u_{n+1}+u_n)^2}\right)c_1(-1)^{n+1} + \frac{1}{(u_{n+1}+u_n)^2}c_1(-1)^n = 0,$$
$$c_1(-1)^n\left(1-1-\frac{1}{(u_{n+1}+u_n)^2}+\frac{1}{(u_{n+1}+u_n)^2}\right) = 0,$$

which is true for every $c_1 \in \mathbb{R}$, so $\mathcal{K}(n,u_n) = c_1(-1)^n$ for each $c_1 \in \mathbb{R}$, and that is the general solution for LSC of the given difference equation.

Remark 4.1. ([1], Note on page 58) In each of the previous examples, we have eliminated w, then u_{n+1}, to leave a difference equation for $\mathcal{K}(n,u_n)$. In general, this is a good approach, but it is not always the simplest way to derive a difference equation for the characteristic. For some difference equations, the calculations are simpler if one eliminates $\mathcal{K}(n,u_n)$ in order to find a difference equation by $\mathcal{K}(n+i,u_{n+i})$, for special $i \geq 1$.

5. Reducing the order of nonlinear difference equations

It is well known that if we want to reduce the order of a linear differential equation or difference equation of the k^{th} order, it is necessary to know a non-trivial solution of the corresponding homogeneous equation. Sophus Lie extended this method to the case of nonlinear differential equations by exploiting one-parameter Lie groups of point symmetries. It turns out that this method can be successfully applied in the case of nonlinear difference equations by determining the compatible canonical coordinate.

First, let us apply the corresponding Lie method to the second-order difference equation

$$u_{n+2} = w\left(n, u_n, u_{n+1}\right). \tag{5.1}$$

Let us assume that we managed to determine the characteristic $K(n;u_n)$ for the given difference equation (5.1) and that s_n is a compatible canonical coordinate. Let

$$r_n = s_{n+1} - s_n = \int \frac{du_{n+1}}{\mathcal{K}\left(n+1, u_{n+1}\right)} - \int \frac{du_n}{\mathcal{K}\left(n, u_n\right)}. \tag{5.2}$$

By using the shift operator on (5.2), we get

$$r_{n+1} = \int \frac{du_{n+2}}{\mathcal{K}\left(n+2, u_{n+2}\right)} - \int \frac{du_{n+1}}{\mathcal{K}\left(n+1, u_{n+1}\right)}. \tag{5.3}$$

On solutions of the difference equation (5.1), we can replace u_{n+2} in (5.3) by w and treat r_{n+1} as a function of n, u_n i u_{n+1}. Then, we obtain

$$\begin{aligned} \frac{\partial r_{n+1}}{\partial u_{n+1}} &= \frac{D_2 w}{\mathcal{K}\left(n+2, u_{n+2}\right)} - \frac{1}{\mathcal{K}\left(n+1, u_{n+1}\right)} = -\frac{D_1 w \mathcal{K}\left(n, u_n\right)}{\mathcal{K}\left(n+1, u_{n+1}\right) \mathcal{K}\left(n+2, w\right)} \\ &= -\frac{\mathcal{K}\left(n, u_n\right)}{\mathcal{K}\left(n+1, u_{n+1}\right)} \frac{\partial r_{n+1}}{\partial u_n}. \end{aligned} \tag{5.4}$$

If we now consider r_{n+1} as a function of n, s_n i s_{n+1}, then

$$\frac{\partial r_{n+1}}{\partial s_{n+1}} = \frac{\partial r_{n+1}}{\partial u_{n+1}} \frac{\partial u_{n+1}}{\partial s_{n+1}} = -\frac{\mathcal{K}\left(n, u_n\right)}{\mathcal{K}\left(n+1, u_{n+1}\right)} \frac{\partial r_{n+1}}{\partial u_n} \frac{\partial u_{n+1}}{\partial s_{n+1}}.$$

From the fact that $s_{n+1} = \int \frac{du_{n+1}}{\mathcal{K}\left(n+1, u_{n+1}\right)}$, it follows $\frac{\partial u_{n+1}}{\partial s_{n+1}} = \mathcal{K}\left(n+1, u_{n+1}\right)$, so we have that

$$\frac{\partial r_{n+1}}{\partial s_{n+1}} = -\mathcal{K}\left(n, u_n\right) \frac{\partial r_{n+1}}{\partial u_n}. \tag{5.5}$$

On the other hand, using that $\frac{\partial u_n}{\partial s_n} = \mathcal{K}\left(n, u_n\right)$, we obtain

$$\frac{\partial r_{n+1}}{\partial s_n} = \frac{\partial r_{n+1}}{\partial u_n} \frac{\partial u_n}{\partial s_n} = \mathcal{K}\left(n, u_n\right) \frac{\partial r_{n+1}}{\partial u_n}. \tag{5.6}$$

Adding (5.5) and (5.6), gives

$$\frac{\partial r_{n+1}}{\partial s_{n+1}} + \frac{\partial r_{n+1}}{\partial s_n} = 0. \tag{5.7}$$

Equation (5.7) implies that r_{n+1} depends only on n, s_n and s_{n+1}. Thus, we have reduced the equation (5.1) to a first-order difference equation of the following form:

$$r_{n+1} = F\left(n, r_n\right). \tag{5.8}$$

We can continue the process if the equation (5.1) can be solved. Let its general solution be of the form

$$r_n = f\left(n; c_1\right).$$

Then, the general solution of the equation (5.1) is given by

$$\int \frac{du_n}{\mathcal{K}(n,u_n)} = s_n = \sum_{k=n_0}^{n-1} f(k;c_1) + c_2. \tag{5.9}$$

As s_n is a canonical compatible coordinate, this solution can be inverted (in principle) to yield u_n.

Example 5.1. *([1], Ex. 2.19) The Lie point symmetries of the following difference equation*

$$u_{n+2} = \frac{2u_n u_{n+1}}{u_{n+1}+u_n} \tag{5.10}$$

include symmetries whose characteristic is $\mathcal{K}(n,u_n) = u_n^2$. Use this result to reduce the equation (5.10) to a first-order difference equation, and hence find the general solution of the equation (5.10).

Solution. Equation (5.10) also appears in [3] as a special case and is reduced to linear equation by substituting $z_n = 1/u_n$. However, using Lie's method, we will demonstrate a slightly more detailed solution than in [1]. From (5.2) let

$$r_{n+1} = \int \frac{du_{n+1}}{u_{n+1}^2} - \int \frac{du_{n+1}}{u_n^2} = \frac{1}{u_n} - \frac{1}{u_{n+1}},$$

and it is obvious that $s_n = -\dfrac{1}{u_n}$ is compatible. Then, based on the solutions of (5.10),

$$r_{n+1} = \frac{1}{u_{n+1}} - \frac{1}{u_{n+2}} = \frac{1}{u_{n+1}} - \frac{u_{n+1}+u_n}{2u_n u_{n+1}} = \frac{1}{2u_{n+1}} - \frac{1}{2u_n} = -\frac{r_n}{2}, \tag{5.11}$$

from which $r_n = c_1\left(-\dfrac{1}{2}\right)^n$. Since $r_n = s_{n+1} - s_n = \Delta s_n$, then

$$s_n = \sum_{k=0}^{n-1} r_k + c_2 = c_1 \sum_{k=0}^{n-1}\left(-\frac{1}{2}\right)^k + c_2 = \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} + c_2 = \frac{2c_1}{3}\left(1-\left(-\frac{1}{2}\right)^n\right) + c_2.$$

From $s_n = -\dfrac{1}{u_n}$ we obtain the general solution of (5.10):

$$u_n = \frac{1}{C_1\left(-\frac{1}{2}\right)^n + C_2}, \quad (C_1, C_2 - \text{new constants}).$$

Remark 5.1. The above reduction process is a generalization of order reduction methods for linear difference equations.

To see the correctness of the previous remark, consider the following second-order linear difference equation:

$$u_{n+2} + a_1(n)\,u_{n+1} + a_0(n)\,u_n = b_n. \tag{5.12}$$

The following is a detailed explanation of all the steps in contrast to the summary overview in [1]. Let us determine the characteristic $\mathcal{K}(n,u_n)$ symmetry of the Lie point for the equation (5.12) using the already described algorithm for the case of a second-order difference equation. Here, since $w(n,u_n,u_{n+1}) = b_n - a_1(n)u_{n+1} - a_0(n)u_n$, the LSC form is

$$\mathcal{K}(n+2,w) + a_1(n)\mathcal{K}(n+1,u_{n+1}) + a_0(n)\mathcal{K}(n,u_n) = 0, \tag{5.13}$$

because $D_1 w = -a_0(n)$ i $D_2 w = -a_1(n)$. Then, due to the elimination of the first term in (5.13)), we have

$$\left(\frac{1}{D_1 w}\frac{\partial}{\partial u_n} - \frac{1}{D_2 w}\frac{\partial}{\partial u_{n+1}}\right)\mathcal{K}(n+2, b_n - a_1(n)u_{n+1} - a_0(n)u_n) = 0,$$

that is,

$$\left(-\frac{1}{a_0(n)}\frac{\partial}{\partial u_n} - \frac{1}{a_1(n)}\frac{\partial}{\partial u_{n+1}}\right)\left(-a_1(n)\mathcal{K}(n+1,u_{n+1}) - a_0(n)\mathcal{K}(n,u_n)\right) = 0,$$

thus,

$$-\frac{1}{a_0(n)}(-a_0(n))\mathcal{K}'(n,u_n) + \frac{1}{a_1(n)}(-a_1(n))\mathcal{K}'(n+1,u_{n+1}) = 0.$$

Applying the operator $\dfrac{d}{du_n}$ to the last equation, we get $\mathcal{K}''(n,u_n) = 0$, which implies that $\mathcal{K}(n,u_n) = c_1 n + c_2$. Taking $c_1 = 1,\ c_2 = 0$ and $u_n = f(n)$, where $f(n)$ is a non-zero solution of the associated homogeneous equation (5.12), we get $\mathcal{K}(n,u_n) = f(n)$. Note that:

$$f(n+2) + a_1(n)f(n+1)a_0(n)f(n) = 0. \tag{5.14}$$

Further, we have

$$\begin{aligned} r_n = s_{n+1} - s_n &= \int \frac{du_{n+1}}{\mathcal{K}(n+1,u_{n+1})} - \int \frac{du_n}{\mathcal{K}(n,u_n)} = \int \frac{du_{n+1}}{f(n+1)} - \int \frac{du_n}{f(n)} \\ &= \frac{1}{f(n+1)}\int du_{n+1} - \frac{1}{f(n)}\int du_n = \frac{u_{n+1}}{f(n+1)} - \frac{u_n}{f(n)}, \end{aligned}$$

so that,

$$\begin{aligned} r_{n+1} &= \frac{u_{n+2}}{f(n+2)} - \frac{u_{n+1}}{f(n+1)} = \frac{b(n) - a_1(n)u_{n+1} - a_0(n)u_n}{f(n+2)} - \frac{u_{n+1}}{f(n+1)} \\ &= \frac{b(n)}{f(n+2)} - \left(a_1(n)\frac{u_{n+1}}{f(n+2)} + a_0(n)\frac{u_n}{f(n+2)} + \frac{u_{n+1}}{f(n+1)}\right) \\ &= \frac{b(n)}{f(n+2)} - \left(a_1(n)\frac{u_{n+1}}{f(n+1)}\frac{f(n+1)}{f(n+2)} + a_0(n)\frac{u_n}{f(n)}\frac{f(n)}{f(n+2)} + \frac{u_{n+1}}{f(n+1)}\right) \\ &= \frac{b(n)}{f(n+2)} - \left[\frac{u_{n+1}}{f(n+1)}\left(a_1(n)\frac{f(n+1)}{f(n+2)} + 1\right) + a_0(n)\frac{u_n}{f(n)}\frac{f(n)}{f(n+2)}\right] \\ &= \frac{b(n)}{f(n+2)} - \left[\frac{u_{n+1}}{f(n+1)}\left(\frac{-f(n+2) - a_0(n)f(n)}{f(n+2)} + 1\right) + a_0(n)\frac{u_n}{f(n)}\frac{f(n)}{f(n+2)}\right] \\ &= \frac{b(n)}{f(n+2)} + a_0(n)\frac{u_{n+1}}{f(n+1)}\frac{f(n)}{f(n+2)} - a_0(n)\frac{u_n}{f(n)}\frac{f(n)}{f(n+2)}\Bigg] \\ &= \frac{b(n)}{f(n+2)} + a_0(n)\frac{f(n)}{f(n+2)}\left(\frac{u_{n+1}}{f(n+1)} - \frac{u_n}{f(n)}\right). \end{aligned}$$

Thus,

$$r_{n+1} = \frac{b(n)}{f(n+2)} + \frac{a_0(n) f(n)}{f(n+2)} r_n,$$

which is a first-order linear difference equation whose general solution is given by

$$r_n = \left(\prod_{i=0}^{n-1} \frac{a_0(i) f(i)}{f(i+2)}\right) r_0 + \sum_{k=0}^{n-1} \left(\prod_{i=k+1}^{n-1} \frac{a_0(i) f(i)}{f(i+2)}\right) \frac{b(k)}{f(k+2)}.$$

Example 5.2. *([1], Example 2.21 - modified here into a more complex problem) Determine the characteristic of Lie point symmetries for the difference equation*

$$u_{n+2} = \frac{2u_{n+1} - u_n + u_n u_{n+1}^2}{1 - u_{n+1}^2 + 2u_n u_{n+1}}. \tag{5.15}$$

Use this characteristic for the reduction of order and solve the difference equation (5.15).

Solution. For the difference equation (5.15), we have

$$D_1 w = \frac{\partial w}{\partial u_n} = \frac{\left(-1 + u_{n+1}^2\right)\left(1 - u_{n+1}^2 + 2u_n u_{n+1}\right) - 2u_{n+1}\left(2u_{n+1} - u_n + u_n u_{n+1}^2\right)}{\left(1 - u_{n+1}^2 + 2u_n u_{n+1}\right)^2}$$

$$= -\frac{\left(1 + u_{n+1}\right)^2}{\left(1 - u_{n+1}^2 + 2u_n u_{n+1}\right)^2},$$

$$D_2 w = \frac{\partial w}{\partial u_{n+1}} = \frac{\left(2 + 2u_n u_{n+1}\right)\left(1 - u_{n+1}^2 + 2u_n u_{n+1}\right) - \left(-2u_{n+1} + 2u_n\right)\left(2u_{n+1} - u_n + u_n u_{n+1}^2\right)}{\left(1 - u_{n+1}^2 + 2u_n u_{n+1}\right)^2}$$

$$= \frac{2\left(1 + u_n^2\right)\left(1 + u_{n+1}^2\right)}{\left(1 - u_{n+1}^2 + 2u_n u_{n+1}\right)^2}.$$

From here, we get $\eta = \ln\left|\frac{D_2 w}{D_1 w}\right| = \ln \frac{2\left(1 + u_n^2\right)}{1 + u_{n+1}^2}$, so

$$D_1 \eta = \frac{1 + u_{n+1}^2}{2\left(1 + u_n^2\right)} \frac{4u_n}{1 + u_{n+1}^2} = \frac{2u_n}{1 + u_n^2} \Longrightarrow D_{12}\eta = 0,$$

$$D_2 \eta = \frac{1 + u_{n+1}^2}{2\left(1 + u_n^2\right)} \frac{-2u_{n+1} 2\left(1 + u_n^2\right)}{\left(1 + u_{n+1}^2\right)^2} = \frac{-2u_{n+1}}{1 + u_{n+1}^2}.$$

Since here $D_{12}\eta = 0$ and $D_2\eta = -ED_1\eta$, we can apply the following formula

$$\mathcal{K}'(n, u_n) - D_1 \eta \mathcal{K}(n, u_n) = \alpha(n),$$

where $\alpha(n) = c_1$. If we take $c_1 = 0$, we get

$$\mathcal{K}'(n,u_n) - \frac{2u_n}{1+u_n^2}\mathcal{K}(n,u_n) = 0,$$

that is,

$$\frac{d\mathcal{K}(n,u_n)}{\mathcal{K}(n,u_n)} = \frac{2u_n}{1+u_n^2}du_n,$$

from which

$$\mathcal{K}(n,u_n) = 1+u_n^2.$$

By using (5.2), we obtain

$$\begin{aligned} r_n = s_{n+1} - s_n &= \int \frac{du_{n+1}}{\mathcal{K}(n+1,u_{n+1})} - \int \frac{du_n}{\mathcal{K}(n,u_n)} = \int \frac{du_{n+1}}{1+u_{n+1}^2} - \int \frac{du_n}{1+u_n^2} \\ &= \arctan u_{n+1} - \arctan u_n = \arctan \frac{u_{n+1}-u_n}{1+u_n u_{n+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} r_{n+1} &= \arctan u_{n+2} - \arctan u_{n+1} = \arctan \frac{u_{n+2}-u_{n+1}}{1+u_{n+1}u_{n+2}} = \arctan \frac{\frac{2u_{n+1}-u_n+u_n u_{n+1}^2}{1-u_{n+1}^2+2u_n u_{n+1}} - u_{n+1}}{1+u_{n+1}\frac{2u_{n+1}-u_n+u_n u_{n+1}^2}{1-u_{n+1}^2+2u_n u_{n+1}}} \\ &= \frac{u_{n+1}-u_n-u_n u_{n+1}^2+u_{n+1}^3}{1+u_{n+1}^2+u_n u_{n+1}+u_n u_{n+1}^3} = \arctan \frac{(u_{n+1}-u_n)\left(1+u_{n+1}^2\right)}{\left(1+u_{n+1}^2\right)(1+u_n u_{n+1})}, \end{aligned}$$

that is,

$$r_{n+1} = \arctan \frac{u_{n+1}-u_n}{1+u_n u_{n+1}} = r_n \qquad (n = 0,1,2,\ldots),$$

so $r_n = c_1$ (c_1 is a constant). Since $r_n = s_{n+1} - s_n = \Delta s_n$, we obtain $s_n = \Delta^{-1}c_1 = c_1 n + c_2$ (c_2 is a constant). It implies that

$$s_n = \arctan u_n \Longrightarrow \arctan u_n = c_1 n + c_2,$$

that is, $u_n = \tan(c_1 n + c_2)$, taking care that the domain of the function arctan is the interval $(-\pi/2, \pi/2)$.

Remark 5.2. The consideration carried out for the equation (5.12) shows that if a particular characteristic is common to a class of difference equations, every equation in the class can be reduced by the same r_n (subject to the canonical coordinate s being compatible with the difference equation). This is also true for ordinary difference equations; most methods for solving a particular class of ordinary differential equations exploit a Lie symmetry group that is shared by all differential equations in that class.

Reduction of order is not restricted to second-order difference equations. By the same process a difference equation of the p^{th}-order with a known non-zero characteristic $\mathcal{K}(n,u_n)$ reduces to a $(p-1)^{th}$-order difference equation, for $r_n = s_{n+1} - s_n$, where s is any compatible canonical coordinate. It is easy to check that LSC yields

$$\frac{\partial r_{n+p-1}}{\partial s_{n+p-1}} + \ldots + \frac{\partial r_{n+p-1}}{\partial s_{n+1}} + \frac{\partial r_{n+p-1}}{\partial s_n} = 0,$$

and consequently there exists a function F such that

$$r_{n+p-1} = F\left(n, s_{n+1} - s_n, \ldots, s_{n+p-1} - s_{n+p-2}\right) = F\left(n, r_n, \ldots, r_{n+p-2}\right). \quad (5.16)$$

If the general solution of the equation (5.16) is of the form

$$r_n = f\left(n; c_1, c_2, \ldots, c_{p-1}\right),$$

the general solution of the original p^{th}-order difference equation is

$$\int \frac{du_n}{\mathcal{K}(n, u_n)} = s_n = \sum_{k=n_0}^{n-1} f\left(k; c_1, c_2, \ldots, c_{p-1}\right) + c_p.$$

Since s is compatible, this solution defines u_n uniquely (for given c_1).

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