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IMPROVING THE SDA ALGORITHMS FOR SOLVING THE T-PALINDROMIC QEPS

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Dedicated to the 75th birthday of our dear Professor Mirjana Vuković

ABSTRACT. The T-palindromic quadratic eigenvalue problem (QEP) $(\lambda^2 B + \lambda C + A)x = 0$, with $A, B, C \in \mathbb{C}^{n\times n}$, $C^T = C$ and $B^T = A$, belongs to the structured quadratic eigenvalue problems. These problems occur in solving fast train vibration problems. Vibration is produced from the interaction between the wheels of trains and the rails underneath. To solve these problems finite element packages can not be used, because of poor accuracy. Standard methods for solving the T-palindromic QEPs are SDAs (the structure-preserving doubling algorithms) methods. The structure of the T-palindromic QEPs allows the improvement of SDAs methods.

1. INTRODUCTION

The T-palindromic quadratic eigenvalue problem (QEP)

$$(\lambda^2 B + \lambda C + A)x = 0, \ x \neq 0, \tag{1.1}$$

with $A, B, C \in \mathbb{C}^{n\times n}$, $C^T = C$ and $B^T = A$, belongs to the structured quadratic eigenvalue problems. For simpler notation, equation (1.1) can be written in the following form

$$(\lambda^2 A_1 + \lambda A_0 + A_1^T)x = 0, \ x \neq 0, \tag{1.2}$$

where $A_0^T = A_0$. Let us define

$$P(\lambda) := \lambda^2 A_1 + \lambda A_0 + A_1^T. \tag{1.3}$$

These problems occur in solving fast train vibration problems. Vibration is produced from the interaction between the wheels of trains and the rails underneath. Matrices A_0 and A_1 are dependent on some parameter ω associated with the speed of the train. The eigenvalues λ are related to the vibration frequencies and the corresponding eigenvectors x reflect the shape of the vibration [5, 11]. Palindromic eigenvalue problems are also used in many other applications such as surface acoustic wave filters, which have wide application in the telecommunication industry. Additional applications of the palindromic eigenvalue problems are given in the papers [1, 13]. The special structure of the palindromic eigenvalue problem (1.2) gives specific spectrum properties that we

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will look back at. Transposing (1.2) implies an important reciprocity property of the spectrum of the palindromic eigenvalue problem,

$$\lambda \in \sigma(P(\lambda)) \Rightarrow \frac{1}{\lambda} \in \sigma(P(\lambda)), \tag{1.4}$$

with $\sigma(\cdot)$ denoting the spectrum, and the convention that 0 and ∞ are considered to be mutually reciprocal. The polynomials

$$\lambda^2 A_1 + \lambda A_0 + A_1^T$$

and

$$\nu^2 A_1 - \nu A_0 + A_1^T$$

define the same palindromic eigenvalue problem ($\nu = -\lambda$).

Spectral symmetries for various types of palindromic eigenvalue problems are given in the paper [13].

An excellent overview of the methods for solving the eigenvalue problems is given in the papers [5, 6].

Finite element packages can not be used to solve these problems, due to their poor accuracy. There are two well-known tools in literature for solving QEPs, linearization and methods based on variational characterization. Applying linearization, we obtain a generalized eigenvalue problem to which the QZ algorithm can be applied. The disadvantage of linearization is that the QZ algorithm does not preserve the palindromic structure. To avoid this problem, in papers [2, 8, 13–16] a palindromic linearization of the form

$$\lambda Z + Z^T \tag{1.5}$$

with

$$Z = \begin{bmatrix} A_1^T & A_0 - A_1 \\ A_1^T & A_1^T \end{bmatrix} \tag{1.6}$$

was presented.

Standard methods for solving the T-palindromic QEPs are SDA (the structure preserving doubling algorithms) methods [5]. The structure of the T-palindromic QEPs allows improvement of SDA methods. In this paper, we will deal with the improvement of this algorithm and its numerical stabilization.

The paper is organized in the following way: In Section 2 the basic idea of SDA algorithms (SDA1 and SDA2) and deflation will be presented. In Section 3 new results related to the stability of the algorithm will be considered. In Section 4 we applied some properties of the T- quadratic eigenvalue problem in order to stabilize the algorithm. The conclusion is given in Section 5.

2. Structure preserving algorithms (SDA)

For greater transparency of the paper we will separate this section into two subsections: Deflation and SDA algorithms.

2.1. **Deflation**

We have seen that eigenvalues 0 and ∞ come in pairs in palindromic eigenvalue problems. Numerical experiments in [5] show that there are below 1.5 % finite nonzero eigenvalues whose absolute value belongs to the segment $[10^{-14}, 10^{14}]$. Therefore deflation has great significance and let us look at it first.

The idea of deflation is to find infinite eigenvalues or those equal to 0, that are a consequence of the singularity of the matrix A_1, and to suppress them (deflation) before applying methods for calculating eigenvalues.

From the mathematical and physical model it is obtained that matrices A_1 and A_0 have the following form:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ L & 0 & 0 \end{bmatrix} \in \mathbb{C}^{n\times n}, \ A_0 = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12}^T & C_{22} & C_{23} \\ 0 & C_{23}^T & C_{33} \end{bmatrix} \in \mathbb{C}^{n\times n},$$

where

$$L \in \mathbb{C}^{n_m\times n_1}, \ C_{11} = C_{11}^T \in \mathbb{C}^{n_1\times n_1}, \ C_{33} = C_{33}^T \in \mathbb{C}^{n_m\times n_m}, C_{22} = C_{22}^T \in \mathbb{C}^{l\times l}$$

and

$$l = n - n_1 - n_m.$$

Assume that C_{22} is nonsingular. Let

$$\Theta = \begin{bmatrix} I_{n_1} & -C_{12}C_{22}^{-1} & 0 \\ 0 & I_l & 0 \\ 0 & -C_{23}^T C_{22}^{-1} & I_{n_m} \end{bmatrix}, \ \Pi = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n_m} \\ 0 & I_l & 0 \end{bmatrix}.$$

Using a similarity transformation, $P(\lambda)$ can be transferred to the following form

$$\begin{aligned} \Pi\Theta P(\lambda)\Theta^T\Pi^T &= \begin{bmatrix} \lambda(C_{11} - C_{12}C_{22}^{-1}C_{12}^T) & L^T - \lambda C_{12}C_{22}^{-1}C_{23} & 0 \\ \lambda(\lambda L - C_{23}^T C_{22}^{-1} C_{12}^T) & \lambda(C_{33} - C_{23}^T C_{22}^{-1} C_{23}) & 0 \\ 0 & 0 & \lambda C_{22} \end{bmatrix} \\ &= \text{diag}\,(I_{n_1}, \lambda I_{n_m}, I_l) \begin{bmatrix} S(\lambda) & 0 \\ 0 & \lambda C_{22} \end{bmatrix}, \end{aligned}$$

where

$$S(\lambda) = \begin{bmatrix} \lambda\tilde{C}_{11} & L^T - \lambda\tilde{C}_{12} \\ \lambda L - \tilde{C}_{12}^T & \tilde{C}_{22} \end{bmatrix}$$

and

$$\begin{aligned} \tilde{C}_{11} &\equiv C_{11} - C_{12}C_{22}^{-1}C_{12}^T, \\ \tilde{C}_{12} &\equiv C_{12}C_{22}^{-1}C_{23}, \\ \tilde{C}_{22} &\equiv C_{33} - C_{23}^T C_{22}^{-1} C_{23}. \end{aligned}$$

Lemma 2.1. *Let* $[x^T, y^T]^T$ *be an eigenvector of* $S(\lambda)$*. Then*

$$\Theta_1^T \Pi_2^T \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ -C_{22}^{-1}(C_{12}^T x + C_{23} y) \\ y \end{bmatrix}$$

is an eigenvector of $P(\lambda)$*. Furthermore,* $\sigma(P(\lambda)) = \sigma(S(\lambda)) \cup \{0, \infty\}$*.*

2.2. **SDA algorithms**

It is important to preserve the palindromic structure at all times. Therefore we use the well-known SDA1 and SDA2 algorithms. Let us look at the SDA1 algorithm. The equation (1.2) can be written in a factored form. By replacing the row-blocks, we obtain that the pencil $S(\lambda)$ is equivalent to

$$\lambda \begin{bmatrix} L & 0 \\ \tilde{C}_{11} & -\tilde{C}_{12} \end{bmatrix} + \begin{bmatrix} -\tilde{C}_{12}^T & \tilde{C}_{22} \\ 0 & L^T \end{bmatrix}, \tag{2.1}$$

which is a generalized standard symplectic form. The structure-preserving doubling algorithm SDA1 can then be applied to solve the corresponding eigenvalue problem.

Let us introduce now the SDA2 algorithm. The T-palindromic pencil for nonsingular X can be written in the following form:

$$P(\lambda) = (\lambda A_1 - X)X^{-1}(\lambda X - A_1^T) + A_1 X^{-1} A_1^T + X + A_0. \tag{2.2}$$

The main idea is to write the T-palindromic pencil (2.2) in the factored form:

$$P(\lambda) = (\lambda A_1 - X)X^{-1}(\lambda X - A_1^T) \tag{2.3}$$

for some nonsingular X if and only if X satisfies the following nonlinear matrix equation with the plus sign:

$$A_1 X^{-1} A_1^T + X + A_0 = 0. \tag{2.4}$$

We apply the SDA algorithm on the equation (2.4), which preserves the structure of the problem [4, 7, 9, 10, 12].

Assume that the matrix $\tilde{C}_{22}$ is invertible. Define $\tilde{S}(\lambda)$ as follows:

$$\begin{aligned} \tilde{S}(\lambda) \;&\equiv\; \begin{bmatrix} I_{n_1} & -L^T \tilde{C}_{22}^{-1} \\ 0 & I_{n_3} \end{bmatrix} S(\lambda) \begin{bmatrix} I_{n_1} & 0 \\ \tilde{C}_{22}^{-1} \tilde{C}_{12}^T & I_{n_3} \end{bmatrix} \\ &= \begin{bmatrix} \lambda(\tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T) + L^T \tilde{C}_{22}^{-1} \tilde{C}_{12}^T & -\lambda \tilde{C}_{12} \\ \lambda L & \tilde{C}_{22} \end{bmatrix} \end{aligned} \tag{2.5}$$

and let $[\tilde{x}^T, \tilde{y}^T]^T$ be an eigenvector of $\tilde{S}(\lambda)$, i.e.

$$\lambda[(\tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12}\tilde{C}_{22}^{-1}\tilde{C}_{12}^T)\tilde{x} - \tilde{C}_{12}\tilde{y}] + L^T \tilde{C}_{22}^{-1}\tilde{C}_{12}^T\tilde{x} = 0, \tag{2.6}$$

$$\lambda L\tilde{x} + \tilde{C}_{22}\tilde{y} = 0. \tag{2.7}$$

Since the matrix $\tilde{C}_{22}$ is invertible, from (2.7) $\tilde{y}$ can be represented as

$$\tilde{y} = -\lambda \tilde{C}_{22}^{-1} L\tilde{x}. \tag{2.8}$$

Let us denote

$$A_{d_1} = \tilde{C}_{12}\tilde{C}_{22}^{-1}L,$$
$$A_{d_0} = \tilde{C}_{11} - L^T\tilde{C}_{22}^{-1}L - \tilde{C}_{12}\tilde{C}_{22}^{-1}\tilde{C}_{12}^T.$$

Suppose that X is nonsingular. Rewrite $P_d(\lambda)$ as

$$P_d(\lambda) = (\lambda A_{d_1} - X)X^{-1}(\lambda X - A_{d_1}^T) + \lambda(A_{d_1}X^{-1}A_{d_1}^T + X + A_{d_0}).$$

Let us apply (2.3) and (2.4) on $P_d(\lambda)$. It follows that $P_d(\lambda)$ can be factorized (or square-rooted) as

$$P_d(\lambda) = (\lambda A_{d_1} - X)X^{-1}(\lambda X - A_{d_1}^T),$$

for some nonsingular X if and only X satisfies the following nonlinear matrix equation with the plus sign:

$$A_{d_1}X^{-1}A_{d_1}^T + X + A_{d_0} = 0.$$

Algorithm 2.1. *(SDA for Palindromic QEP)*
Input: $C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, L; \tau$ *(a small tolerance)*;
Output: an eigenpar $(\lambda, [x^T, z^T, y^T]^T)$ *of Palindromic QEP.*
Compute
$\tilde{C}_{11} = C_{11} - C_{12}C_{22}^{-1}C_{12}^T,$
$\tilde{C}_{12} = C_{12}C_{22}^{-1}C_{23},$
$\tilde{C}_{22} = C_{33} - C_{23}^T C_{22}^{-1}C_{23},$
$A_{d_1} = \tilde{C}_{12}\tilde{C}_{22}^{-1}L,$
$A_{d_0} = \tilde{C}_{11} - L^T\tilde{C}_{22}^{-1}L - \tilde{C}_{12}\tilde{C}_{22}^{-1}\tilde{C}_{12}^T;$
Set $k = 0, R_k = A_{d_1}^T, Q_k = -A_{d_0}$ *and* $P_k = 0$;
Do until convergence:
Compute $R_{k+1} = R_k(Q_k - P_k)^{-1}R_k,$
$Q_{k+1} = Q_k - R_k^T(Q_k - P_k)^{-1}R_k,$
$P_{k+1} = P_k + R_k(Q_k - P_k)^{-1}R_k^T,\ k = k+1;$
If $\|Q_k - Q_{k-1}\| \leqslant \tau\|Q_k\|$, *Stop* ;
End;
Compute the left/right eigenpairs $(\lambda_u, \hat{x}_s),\ (\lambda_u, \hat{x}_r)$ *of* $Q_k\hat{x} = \lambda A_{d_1}\hat{x}$;
Solve $(\lambda_u Q_k - A_{d_1}^T)\tilde{x}_u = Q_k\hat{x}_r$;
Set $\lambda_s = \lambda_u^{-1}$;
Solve $\tilde{C}_{22}\tilde{y} = -\lambda L\tilde{x}$ *with* $(\lambda, \tilde{x}) = (\lambda_s, \tilde{x}_s)$ *or* $(\lambda, \tilde{x}) = (\lambda_u, \tilde{x}_u)$;
Set $x = \tilde{x}$; *Compute* $y = \tilde{C}_{22}^{-1}\tilde{C}_{12}^T\tilde{x} + \tilde{y},\ z = -C_{22}^{-1}(C_{12}^T x + C_{23}y)$;

3. New ideas for improving the stability of the algorithm

In addition to preserving the structure of the eigenvalue problem during the implementation of the algorithm, it is very important to pay attention to the stability of the algorithm. The analysis of Algorithm 2.1 clearly shows the points at which the problem may occur:

- invertibility of the matrix $Q_k - P_k$ in the Algorithm 2.1;
- efficient preprocessing of problems when C_{22} is ill-conditioned;
- efficient preprocessing of problems when L is ill-conditioned.

3.1. **Invertibility of significant matrices**

After deflation, the palindromic eigenvalue problem

$$(\lambda^2 A_{d_1} + \lambda A_{d_0} + A_{d_1}^T)v = 0, \ v \neq 0 \tag{3.1}$$

is considered, which is a lower dimension problem.

It is clear that the singularity of the matrix A_{d_0} can cause a problem in Algorithm 2.1, because for $k = 0$, $Q_0 = -A_{d_0}$, $P_0 = 0$, and the invertibility of the matrix $Q_0 - P_0$ in the first step of Algorithm 2.1 is required.

Let

$$A_{d_0} = QA_{d_0}^{'}Q^T$$

be the Schur decomposition of a matrix A_{d_0}. Then $A_{d_0}^{'}$ is a diagonal matrix. Elements of the matrix $A_{d_0}^{'}$ are eigenvalues of the matrix A_{d_0}. Matrices Q and Q^T are orthogonal. Without loss of generality let us assume that the first p elements on the diagonal of the matrix $A_{d_0}^{'}$ are zeros.

The equation (3.1) can be written in the following form

$$(\lambda^2 A_{d_1} + \lambda QA_{d_0}^{'}Q^T + A_{d_1}^T)v = 0. \tag{3.2}$$

After multiplying (3.2) with Q^T on the left we obtain an equivalent eigenvalue problem

$$(\lambda^2 Q^T A_{d_1} Q + \lambda A_{d_0}^{'} + Q^T A_{d_1}^T Q)Q^T v = 0, \tag{3.3}$$

which is the T-palindromic eigenvalue problem, with eigenvector $w = Q^T v = \begin{bmatrix} a \\ b \end{bmatrix}$.

For simpler notation, the problem can be presented in the following form

$$(\lambda^2 B_{d_1} + \lambda B_{d_0} + B_{d_1}^T)w = 0, \tag{3.4}$$

respectively in the block matrix form

$$\left(\lambda^2 \begin{bmatrix} B_{d_{11}} & B_{d_{12}} \\ B_{d_{21}} & B_{d_{22}} \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & B_{d_{02}} \end{bmatrix} + \begin{bmatrix} B_{d_{11}}^T & B_{d_{21}}^T \\ B_{d_{12}}^T & B_{d_{22}}^T \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{3.5}$$

It is clear that we obtain the quadratic T-palindromic eigenvalue problem

$$(\lambda^2 B_{d_{22}} + \lambda B_{d_{02}} + B_{d_{22}}^T)b = 0, b \neq 0.$$

Suppose that $a = 0$ and that the eigenvector b satisfies

$$(\lambda^2 B_{d_{12}} + B_{d_{21}}^T)b = 0.$$

In this case we obtain the T- palindromic eigenvalue problem of a lower dimension then the one we were dealing with, and w is an eigenvector of the eigenvalue problem (3.5).

It is also clear in the case that a is an eigenvector of the T-palindromic linear eigenvalue problem

$$B_{d_{11}}^T a = -pB_{d_{11}}a,\ p = \lambda^2,$$

and $b = 0$ and the eigenvector a satisfies the additional condition

$$(\lambda^2 B_{d_{21}} + B_{d_{12}}^T)a = 0.$$

Then $w = \begin{bmatrix} a \\ 0 \end{bmatrix}$ is an eigenvector of the T-palindromic quadratic eigenvalue problem (3.4).

If the Schur decomposition of the matrix A does not lead to the T-palindromic quadratic eigenvalue problem of lower dimension or to the linear T-palindromic eigenvalue problem of lower dimension, then in Algorithm 2.1 the singular matrix $(-A_{d_0})^{-1}$ which does not exist needs to be replaced with the pseudoinverse matrix $(-A_{d_0})^{+}$, which is the best approximation of the inverse matrix.

If Q_k and P_k from Algorithm 2.1 have the property that $Q_k - P_k$ is a singular matrix, let us write

$$Q_k - P_k = \bar{Q}_k (Q_k - P_k)' \bar{Q}_k^T,$$

where $(Q_k - P_k)'$ is a diagonal matrix which has eigenvalues of the matrix $(Q_k - P_k)$ and the first s diagonal elements are equal to zero.

Respectively,

$$(Q_k - P_k)' = \begin{bmatrix} 0 & 0 \\ 0 & (Q_k - P_k)'' \end{bmatrix},$$

where $(Q_k - P_k)''$ is an invertible diagonal matrix, which has eigenvalues of the matrix $(Q_k - P_k)$ different from zero. In this case it is suggested to use the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & ((Q_k - P_k)'')^{-1} \end{bmatrix},$$

instead of the matrix $(Q_k - P_k)^{-1}$ which does not exist.

3.2. **ill-conditioned** C_{22}

In Subsection 2.1

$$\tilde{C}_{11} \equiv C_{11} - C_{12}C_{22}^{-1}C_{12}^T,$$
$$\tilde{C}_{12} \equiv C_{12}C_{22}^{-1}C_{23},$$
$$\tilde{C}_{22} \equiv C_{33} - C_{23}^T C_{22}^{-1}C_{23},$$

were defined, where C_{22} is invertible. If the matrix C_{22} is ill-conditioned, the calculation of the inverse matrix C_{22}^{-1} is numerically unstable. In order to stabilize this process QR-factorization with Givens rotation is used. Thus,

$$C_{22} = Q_{22}R_{22},$$
$$C_{22}^{-1} = R_{22}^{-1}Q_{22}^T.$$

It follows that

$$\tilde{C}_{11} \equiv C_{11} - C_{12}R_{22}^{-1}Q_{22}^T C_{12}^T,$$
$$\tilde{C}_{12} \equiv C_{12}R_{22}^{-1}Q_{22}^T C_{23},$$
$$\tilde{C}_{22} \equiv C_{33} - C_{23}^T R_{22}^{-1}Q_{22}^T C_{23}.$$

The algorithm cost is $\frac{4}{3}l^3$ flops.

If QR-factorization shows instability, the matrix C_{22} can be replaced by the pseudoinverse (Moore-Penrose inverse) C_{22}^+.

3.3. **ill-conditioned** L

Numerical results prove that L is ill-conditioned. The conditional $\kappa(L)$ is equal to

$$\kappa(L) = \frac{\max_{\|x\|=1} \|Lx\|}{\min_{\|x\|=1} \|Lx\|} \approx 10^{20}.$$

In previous papers, preconditioning was not performed. For larger dimensions preconditioning must be done and our proposal is to apply QR-factorization of the matrix L.

4. Application of some properties of spectrum in the case of numerical instability of the SDA algorithm

In the previous section, we saw some of the classical ideas that can help to stabilize Algorithm 2.1. In this section we consider some properties of the quadratic palindromic eigenvalue problems, so we can replace the SDA2 algorithm, in the case of some of the above problems, with an appropriate algorithm for the linear eigenvalue problem, in the sense of the following two theorems:

Theorem 4.1. *If* $\pm 1 \notin \sigma(P(\lambda))$ *then the eigenvector* x *of the palindromic eigenvalue problem*

$$(\lambda^2 A_{d_1} + \lambda A_{d_0} + A_{d_1}^T)x = 0,\ A_{d_0}^T = A_{d_0},\ x \neq 0, \tag{4.1}$$

satisfies the equation

$$(\lambda + \frac{1}{\lambda})x^T A_{d_1} x + x^T A_{d_0} x = 0. \tag{4.2}$$

Proof 4.1. *Let us multiply the equation* (4.1) *with* x^T *on the left. We obtain:*

$$\lambda^2 x^T A_{d_1} x + \lambda x^T A_{d_0} x + x^T A_{d_1}^T x = 0. \tag{4.3}$$

Thus for $x^T A_{d_1} x \in \mathbb{C}$,

$$x^T A_{d_1}^T x = (x^T A_{d_1}^T x)^T,$$

$$x^T A_{d_1}^T x = x^T (A_{d_1}^T)^T (x^T)^T = x^T A_{d_1} x. \tag{4.4}$$

From the equation (4.3) *we have*

$$\lambda^2 x^T A_{d_1} x + \lambda x^T A_{d_0} x + x^T A_{d_1} x = 0. \tag{4.5}$$

If λ is an eigenvalue, and x is the right eigenvector of the eigenproblem (4.3) *then $\frac{1}{\lambda}$ is an eigenvalue and x^T is the left eigenvector of the eigenproblem* (4.3)*. This means that*

$$x^T (\frac{1}{\lambda^2} A_{d_1} + \frac{1}{\lambda} A_{d_0} + A_{d_1}^T) = 0. \tag{4.6}$$

If we multiply the equation (4.6) *on the right with the eigenvector x we obtain*

$$\frac{1}{\lambda^2} x^T A_{d_1} x + \frac{1}{\lambda} x^T A_{d_0} x + x^T A_{d_1}^T x = 0. \tag{4.7}$$

According to (4.4) *and* (4.7) *we get*

$$\frac{1}{\lambda^2} x^T A_{d_1} x + \frac{1}{\lambda} x^T A_{d_0} x + x^T A_{d_1} x = 0. \tag{4.8}$$

Subtracting (4.8) *from* (4.5) *we obtain*

$$(\lambda^2 - \frac{1}{\lambda^2}) x^T A_{d_1} x + (\lambda - \frac{1}{\lambda}) x^T A_{d_0} x = 0,$$

respectively

$$(\lambda - \frac{1}{\lambda})((\lambda + \frac{1}{\lambda}) x^T A_{d_1} x + x^T A_{d_0} x) = 0.$$

Since $\lambda \neq \pm 1$,

$$(\lambda + \frac{1}{\lambda}) x^T A_{d_1} x + x^T A_{d_0} x = 0. \tag{4.9}$$

holds.

It is interesting that the following holds:

Theorem 4.2. *If $\pm i \notin \sigma(P(\lambda))$ then the eigenvector x of the palindromic eigenvalue problem*

$$(\lambda^2 A_{d_1} + \lambda A_{d_0} + A_{d_1}^T) x = 0, \; A_{d_0}^T = A_{d_0}, \; x \neq 0, \tag{4.10}$$

satisfies the equation

$$(\lambda + \frac{1}{\lambda}) x^T A_{d_1} x + x^T A_{d_0} x = 0. \tag{4.11}$$

Proof 4.2. *In the Proof 4.1 it is proved that equations* (4.5) *and* (4.8) *hold. By adding these two equations we obtain*

$$(\lambda^2 + 2 + \frac{1}{\lambda^2}) x^T A_{d_1} x + (\lambda + \frac{1}{\lambda}) x^T A_{d_0} x = 0,$$

i.e.

$$(\lambda + \frac{1}{\lambda})((\lambda + \frac{1}{\lambda}) x^T A_{d_1} x + x^T A_{d_0} x) = 0.$$

Since $\lambda \neq \pm i$,

$$(\lambda + \frac{1}{\lambda}) x^T A_{d_1} x + x^T A_{d_0} x = 0$$

holds.

Lemma 4.1. *For the invertible matrix A_{d_1} eigenvector of the linear eigenvalue problem*

$$(A_{d_1}^{-1}A_0)x = \mu x \qquad (4.12)$$

satisfies the equation (4.9)*, where* $\mu = -(\lambda + \frac{1}{\lambda})$.

If matrix A_{d_1} is non-invertible then the eigenvector of the linear generalized eigenvalue problem

$$A_{d_0}x = -\mu A_{d_1}x$$

satisfies the equation (4.9).

Lemma 4.2. *The eigenvalue of the eigenvalue problem* (1.2) *is* 0 *respectively* ∞ *if and only if the matrix A_1 is singular.*

Since the deflation of the palindromic eigenvalue problem was done first, it is clear that if the problem is reduced to a linear eigenvalue problem then it is not a generalized eigenvalue problem. In this case we apply the first part of Lemma 4.1.

Proposition 4.1. *If* $\pm 1 \notin \sigma(P(\lambda))$ *or* $\pm i \notin \sigma(P(\lambda))$*, the eigenvectors of the eigenvalue problem* (4.1) *are obtained as eigenvectors of the linear eigenvalue problem* (4.12) *or as a vector x which is normal to the vector*

$$(\lambda + \frac{1}{\lambda})xA_{d_1}x + A_{d_0}x = 0.$$

Proposition 4.2. *If the eigenvector of the eigenvalue problem* (4.1) *and eigenvalue problem* (4.12) *match, then the eigenvalue λ of the problem* (4.1) *and it's reciprocal eigenvalue $\frac{1}{\lambda}$ are obtained as solutions of the following equation*

$$-\lambda - \frac{1}{\lambda} = \mu,$$

where μ is the eigenvalue of the problem (4.12).

Theorem 4.3. *If the matrix $A_{d_1}^{-1}$ can be diagonalized, then eigenvectors of the eigenvalue problem* (4.1) *are obtained according to Proposition 4.2.*

Proof 4.3. *Since the matrix $A_{d_1}^{-1}$ is diagonalized its eigenvectors are linearly independent and form the base of the space $\mathbb{C}^n$. Thus, only the zero vector is normal to the vector* $(\lambda + \frac{1}{\lambda})xA_{d_1}x + A_{d_0}x = 0$.

Remark 4.1. In the case $\pm 1 \in \sigma(P(\lambda))$ the deflation of eigenvalues is given in the paper [13].

From the above it can be seen that in the case of the problem of the SDA2 algorithm, it is better to try to reduce the problem to a linear eigenvalue problem than to apply classical stabilization methods.

5. Conclusion

In this paper important issues that can affect the convergence and stability of the algorithm are discussed. In previous papers these problems were overcome by the combination of SDA1 and SDA2 algorithms. Significant improvements were obtained. Preconditioning of significant matrices was used, which is numerically better than the combination of SDA1 and SDA2 algorithms. Also, we improved the algorithm using some significant properties of the spectrum of the T- quadratic palindromic eigenvalue problem. Thus the properties of the palindromic pencil and the QEP structure was preserved. Further research: performing more extensive numerical tests, as well as the consideration of problems of higher dimensions and expanding the consideration of spectrum properties.

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