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ASYMPTOTIC BEHAVIOR OF NON-AUTONOMOUS COMPETITIVE SYSTEMS OF DIFFERENCE EQUATIONS

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Dedicated to the 75th birthday of our dear Professor Mirjana Vuković

ABSTRACT. The problem of the behavior (convergence and stability) of the solutions of non-autonomous systems of difference equations with asymptotically constant coefficients is still open. In previous research, the main interest was related to the results on global attractiveness for some classes of non-autonomous competitive and cooperative systems. In this paper, using those results and the same methods in the partial ordering of the space $\mathbb{R}_+^2$, we prove the general theorem for any non-autonomous competitive system with asymptotically constant coefficients. The obtained results are also illustrated with concrete examples.

1. INTRODUCTION

An autonomous system of difference equations has constant coefficients, while a non-autonomous system has variable coefficients (sequences). In this paper, we will consider the behavior of non-autonomous competitive systems of difference equations whose coefficients are asymptotically constant. This problem is still open and has yet to have a general result covering all cases. Nevertheless, we will give such a result here for the case of competitive systems. It relies on previously obtained results for some general classes of these systems.

In [9] the following non-autonomous competitive systems whose coefficients are asymptotically constant:

$$X_{n+1}=\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}=\begin{bmatrix} a_n f(x_n,y_n) \\ b_n g(x_n,y_n) \end{bmatrix},\quad n=0,1,\ldots,$$

$$X_{n+1}=\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}=\begin{bmatrix} \dfrac{x_n}{a_n+y_n} \\ \dfrac{y_n}{b_n+x_n} \end{bmatrix},\quad n=0,1,\ldots,$$

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$$X_{n+1}=\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}=\begin{bmatrix} \dfrac{\alpha_n x_n}{a_n+y_n} \\ \dfrac{\beta_n y_n}{b_n+x_n} \end{bmatrix},\ n=0,1,...,$$

and the following non-autonomous *Leslie-Gower model*

$$X_{n+1}=\begin{bmatrix} \dfrac{a_n x_n}{1+c_n^{(11)}x_n+c^{(12)}y_n} \\ \dfrac{b_n y_n}{1+c^{(21)}x_n+c_n^{(22)}y_n} \end{bmatrix},\quad n=0,1,2,....,$$

were considered, and a theorem of the form Theorem 2.1 was proved in that case. After that, the corresponding Leslie-Gower evolutionary model with two Fisher's equations was considered separately.

Also, in [10] the following non-autonomous cooperative systems:

$$x_{n+1}^{(i)}=A_n^{(i)}x_n^{(i)}\frac{\prod\limits_{i\neq j=1}^{k} x_n^{(j)}}{1+\prod\limits_{i\neq j=1}^{k} x_n^{(j)}},\quad n=0,1,...;i=1,2,...,k,$$

and

$$\left.\begin{aligned} x_{n+1}&=\frac{a_n x_n}{\delta_1+x_n}+\frac{b_n y_n}{\delta_2+y_n}, \\ y_{n+1}&=\frac{c_n x_n}{\delta_2+x_n}+\frac{d_n y_n}{\delta_1+y_n}, \end{aligned}\right\} n=0,1,\ldots$$

were considered.

All obtained results are based on the behavior of the corresponding autonomous competitive and cooperative systems. Regarding autonomous competitive and cooperative systems, see [2–5, 7, 8, 11–15].

In this paper, we use the method of difference inequalities to prove global attractivity results for two-dimensional competitive systems in [9]. The map $F:\mathbb{R}_+^2\to\mathbb{R}_+^2$, $F=(f,g)$ is called a competitive map if f is non-decreasing in the first variable and non-increasing in the second variable, and g is non-increasing in the first variable and non-decreasing in the second variable. However, the results in [9] are two-dimensional, and it is not clear how to extend them to the k-dimensional case for $k>2$.

Also, we will use the so-called "north-east" partial ordering of the space $\mathbb{R}_+^2$, defined it in the following way:

$$X_n=\begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \end{bmatrix}\preccurlyeq_{ne} Y_n=\begin{bmatrix} y_n^{(1)} \\ y_n^{(2)} \end{bmatrix}\Longleftrightarrow\left(x_n^{(1)}\leq y_n^{(1)}\ \text{and}\ x_n^{(2)}\leq y_n^{(2)}\right),$$

and the so-called "south-east" partial ordering of the space $\mathbb{R}_+^2$ defined by

$$X_n=\begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \end{bmatrix}\preccurlyeq_{se} Y_n=\begin{bmatrix} y_n^{(1)} \\ y_n^{(2)} \end{bmatrix}\Longleftrightarrow\left(x_n^{(1)}\leq y_n^{(1)}\ \text{and}\ x_n^{(2)}\geq y_n^{(2)}\right).$$

If we replace "$\leq$" and "$\geq$" with "$<$" and "$>$" in the above relations, then "$\preceq$" also changes to "$\prec$".

2. MAIN RESULTS

In [9], the following lemma is proved.

Lemma 2.1. *([9], Lemma 1) Assume that*

a) $F:\mathbb{R}_+^2\to\mathbb{R}_+^2$, $F=\begin{bmatrix} f\\ g\end{bmatrix}$ *is a competitive map.*

b) $\{X_n\}$, $\{Y_n\}$, $\{Z_n\}$ *are sequences of the real components in* $\mathbb{R}_+^2$ *such that*

$$X_0 \preccurlyeq_{se} Y_0 \preccurlyeq_{se} Z_0$$

and

$$\left.\begin{array}{r} X_{n+1}\preccurlyeq_{se} F(X_n)\\ Y_{n+1}=F(Y_n)\\ Z_{n+1}\succcurlyeq_{se} F(Z_n)\end{array}\right\},\quad n=0,1,2,\ldots .$$

Then,

$$X_n \preccurlyeq_{se} Y_n \preccurlyeq_{se} Z_n,\quad n=0,1,2,\ldots . \tag{2.1}$$

Lemma 2.1 is necessary to obtain the following general result on the behavior of non-autonomous competitive systems of difference equations whose coefficients are asymptotically constant.

Theorem 2.1. *Consider the following non-autonomous system of difference equations*

$$X_{n+1}=\begin{bmatrix} f(a_1(n),\ldots,a_k(n);x_n,y_n)\\ g(b_1(n),\ldots,b_l(n);x_n,y_n)\end{bmatrix},\quad n=0,1,2,\ldots, \tag{2.2}$$

where $A_n=[a_1(n),\ldots,a_k(n),b_1(n),\ldots,b_l(n)]^T$, *k and l are positive integers, and* $F=\begin{bmatrix} f\\ g\end{bmatrix}:\mathbb{R}_+^2\to\mathbb{R}_+^2$ *is a competitive map. Assume that*

$$\lim_{n\to\infty}A_n=\lim_{n\to\infty}[a_1(n),\ldots,a_k(n),b_1(n),\ldots,b_l(n)]^T=[a_1,\ldots,a_k,b_1,\ldots,b_l]^T=A. \tag{2.3}$$

Also, assume that there exists $\varepsilon_0=\left[\varepsilon_0^{(1)},\ldots,\varepsilon_0^{(k)},\varepsilon_0^{(k+1)},\ldots,\varepsilon_0^{(k+l)}\right]^T \succ_{ne} \left[\underbrace{0,\ldots,0}_{k+l}\right]^T$ *such that for every* $A_\varepsilon=[\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_l]^T$, *with*

$$\alpha_i\in\left(a_i-\varepsilon_0^{(i)},a_i+\varepsilon_0^{(i)}\right),\ \beta_j\in\left(b_j-\varepsilon_0^{(j)},b_j+\varepsilon_0^{(j)}\right),\ i=1,\ldots,k;j=1,\ldots,l,$$

all the solutions of the system

$$Y_{n+1}=\begin{bmatrix} f(\alpha_1,\ldots,\alpha_k;u_n,v_n)\\ g(\beta_1,\ldots,\beta_l;u_n,v_n)\end{bmatrix},\quad n=0,1,2,\ldots,;k,l\in\mathbb{Z}^+ \tag{2.4}$$

converge to a constant solution $\overline{Y}_{A_\varepsilon}=\begin{bmatrix}\overline{x}_{A_\varepsilon}\\ \overline{y}_{A_\varepsilon}\end{bmatrix}$.

Additionally, suppose that $\lim\limits_{A_\varepsilon\to A}\overline{Y}_{A_\varepsilon}=\overline{Y}_A$.

Then, every solution of the system (2.2) converges to $\overline{Y}_A$.

Proof. According to (2.3), for any

$$\varepsilon_1 = \left[\varepsilon_{1,1},...,\varepsilon_{1,k}\right]^T \succ_{ne} \left[\underbrace{0,...,0}_{k}\right]^T \text{ and } \varepsilon_2 = \left[\varepsilon_{2,1},...,\varepsilon_{2,l}\right]^T \succ_{ne} \left[\underbrace{0,...,0}_{l}\right]^T,$$

there exists $N = N(\varepsilon_1, \varepsilon_2)$ such that for $n \geq N$ the following holds:

$$\begin{aligned} a_i - \varepsilon_{1,i} &< a_i(n) < a_i + \varepsilon_{1,i}, \quad i = 1,2,...,k, \\ b_j - \varepsilon_{2,j} &< b_j(n) < b_j + \varepsilon_{2,j}, \quad j = 1,2,...l. \end{aligned}$$

Thus, for $n \geq N$, we get

$$\begin{aligned} \begin{bmatrix} f\left(a_{L,1},...,a_{L,k};x_n,y_n\right) \\ g\left(b_{L,1},...,b_{L,l};x_n,y_n\right) \end{bmatrix} \preccurlyeq_{se} X_{n+1} &= \begin{bmatrix} f\left(a_1(n),...,a_k(n);x_n,y_n\right) \\ g\left(b_1(n),...,b_l(n);x_n,y_n\right) \end{bmatrix} \\ &\preccurlyeq_{se} \begin{bmatrix} f\left(a_{D,1},...,a_{D,k};x_n,y_n\right) \\ g\left(b_{D,1},...,b_{D,l};x_n,y_n\right) \end{bmatrix}, \end{aligned} \tag{2.5}$$

for $a_{L,i} = a_i - \varepsilon_{1,i}$, or $a_{L,i} = a_i + \varepsilon_{1,i}$ $(i = 1,...,k)$ and $b_{L,j} = b_j - \varepsilon_{2,j}$ or $b_{L,j} = b_j + \varepsilon_{2,j}$ $(j = 1,...,l)$, and

$$a_{D,i} = \begin{cases} a_i - \varepsilon_{1,i} & \text{if } a_{L,i} = a_i + \varepsilon_{1,i} \\ a_i + \varepsilon_{1,i} & \text{if } a_{L,i} = a_i - \varepsilon_{1,i} \end{cases} \quad (i = 1,...,k),$$

and

$$b_{D,j} = \begin{cases} b_j - \varepsilon_{2,j} & \text{if } b_{L,j} = b_j + \varepsilon_{2,j} \\ b_j + \varepsilon_{2,j} & \text{if } b_{L,j} = b_j - \varepsilon_{2,j} \end{cases} \quad (j = 1,...,l).$$

Since $F = \begin{bmatrix} f\left(a_1(n),...,a_k(n);x_n,y_n\right) \\ g\left(b_1(n),...,b_l(n);x_n,y_n\right) \end{bmatrix}$ is a competitive map, Lemma 2.1 implies

$$L_n \preccurlyeq_{se} X_n \preccurlyeq_{se} U_n, \quad n \geq N(\varepsilon), \tag{2.6}$$

where $\{L_n\} = \left\{\begin{bmatrix} l_n^{(1)} \\ l_n^{(2)} \end{bmatrix}\right\}$ satisfies

$$L_{n+1} = \begin{bmatrix} f\left(a_{L,1},...,a_{L,k};l_n^{(1)},l_n^{(2)}\right) \\ g\left(b_{L,1},...,b_{L,l};l_n^{(1)},l_n^{(2)}\right) \end{bmatrix},$$

and $\{U_n\} = \left\{\begin{bmatrix} u_n^{(1)} \\ u_n^{(2)} \end{bmatrix}\right\}$ satisfies

$$U_{n+1} = \begin{bmatrix} f\left(a_{D,1},...,a_{D,k};u_n^{(1)},u_n^{(2)}\right) \\ g\left(b_{D,1},...,b_{D,l}u_n^{(1)},u_n^{(2)}\right) \end{bmatrix}.$$

By using (2.6), we obtain

$$\lim_{n\to\infty} L_n \preccurlyeq_{se} \liminf_{n\to\infty} X_n \preccurlyeq_{se} \limsup_{n\to\infty} X_n \preccurlyeq_{se} \lim_{n\to\infty} U_n,$$

i.e.,

$$\overline{Y}_{\alpha_{\varepsilon_1,\varepsilon_2}} \preccurlyeq_{se} \liminf_{n\to\infty} X_n \preccurlyeq_{se} \limsup_{n\to\infty} X_n \preccurlyeq_{se} \overline{Y}_{\beta_{\varepsilon_1,\varepsilon_2}}, \tag{2.7}$$

where $\alpha_{\varepsilon_1,\varepsilon_2} = \begin{bmatrix} \mathbf{a}_L \\ \mathbf{b}_L \end{bmatrix}$, $\beta_{\varepsilon_1,\varepsilon_2} = \begin{bmatrix} \mathbf{a}_D \\ \mathbf{b}_D \end{bmatrix}$, and

$$\mathbf{a}_L = \begin{bmatrix} a_{L,1} \\ \vdots \\ a_{L,k} \end{bmatrix}, \mathbf{b}_L = \begin{bmatrix} b_{L,1} \\ \vdots \\ b_{L,l} \end{bmatrix}, \mathbf{a}_D = \begin{bmatrix} a_{D,1} \\ \vdots \\ a_{D,k} \end{bmatrix}, \mathbf{b}_D = \begin{bmatrix} b_{D,1} \\ \vdots \\ b_{D,l} \end{bmatrix}.$$

Since $\lim\limits_{\substack{\varepsilon_1\to\mathbf{0}\\ \varepsilon_2\to\mathbf{0}}} \overline{Y}_{\alpha_{\varepsilon_1,\varepsilon_2}} = \lim\limits_{\substack{\varepsilon_1\to\mathbf{0}\\ \varepsilon_2\to\mathbf{0}}} \overline{Y}_{\beta_{\varepsilon_1,\varepsilon_2}} = \overline{Y}_A$, where $\mathbf{0} = \begin{bmatrix} \underbrace{0,\ldots,0}_{k+l} \end{bmatrix}^T$, (2.7) implies that the sequence $\{X_n\}$ is convergent and that

$$\lim_{n\to\infty} X_n = \overline{Y}_A. \qquad \square$$

Remark 2.1. The condition on the system (2.4) means that the map associated with the system (2.2) is structurally stable.

Now, we will state a more general non-autonomous Leslie-Gower model and demonstrate the individual steps of the proof of Theorem 2.1.

Consider the following general non-autonomous Leslie-Gower model (see [6], [9], [16])

$$X_{n+1} = \begin{bmatrix} \dfrac{a_n x_n}{1 + c_n^{(11)} x_n + c_n^{(12)} y_n} \\ \dfrac{b_n y_n}{1 + c_n^{(21)} x_n + c_n^{(22)} y_n} \end{bmatrix}, \quad n = 0, 1, 2, ..., \tag{2.8}$$

and assume that

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \left[a_n, c_n^{(11)}, c_n^{(12)}, b_n, c_n^{(21)}, c_n^{(22)} \right]^T = \left[a, c^{(11)}, c^{(12)}, b, c^{(21)}, c^{(22)} \right]^T = A.$$

Note that the condition (2.5) for the system (2.8) has the form:

$$L_n \preccurlyeq_{se} X_{n+1} = \begin{bmatrix} \dfrac{a_n x_n}{1 + c_n^{(11)} x_n + c^{(12)} y_n} \\ \dfrac{b_n y_n}{1 + c^{(21)} x_n + c_n^{(22)} y_n} \end{bmatrix} \preccurlyeq_{se} U_n,$$

where

$$L_n = \begin{bmatrix} \dfrac{(a - \varepsilon_{1,1}) x_n}{1 + \left(c^{(11)} + \varepsilon_{1,2}\right) x_n + \left(c^{(12)} + \varepsilon_{1,3}\right) y_n} \\ \dfrac{(b + \varepsilon_{2,1}) y_n}{1 + \left(c^{(21)} - \varepsilon_{2,2}\right) x_n + \left(c^{(22)} - \varepsilon_{2,3}\right) y_n} \end{bmatrix},$$

$$U_n = \begin{bmatrix} \dfrac{(a + \varepsilon_{1,1}) x_n}{1 + \left(c^{(11)} - \varepsilon_{1,2}\right) x_n + \left(c^{(12)} - \varepsilon_{1,3}\right) y_n} \\ \dfrac{(b - \varepsilon_{2,1}) y_n}{1 + \left(c^{(21)} + \varepsilon_{2,2}\right) x_n + \left(c^{(22)} + \varepsilon_{2,3}\right) y_n} \end{bmatrix},$$

and

$$
\begin{aligned}
a-\varepsilon_{1,1} &< a_n < a+\varepsilon_{1,1},\\
c^{(11)}-\varepsilon_{1,2} &< c_n^{(11)} < c^{(11)}+\varepsilon_{1,2},\\
c^{(12)}-\varepsilon_{1,3} &< c_n^{(12)} < c^{(12)}+\varepsilon_{1,3},\\
b-\varepsilon_{2,1} &< b_n < b+\varepsilon_{2,1},\\
c^{(21)}-\varepsilon_{2,2} &< c_n^{(21)} < c^{(21)}+\varepsilon_{2,2},\\
c^{(22)}-\varepsilon_{2,3} &< c_n^{(22)} < c^{(22)}+\varepsilon_{2,3},
\end{aligned}
$$

for $n \geq N$.

By Theorem 2.1 the non-autonomous system (2.8) is asymptotic to the limiting system

$$
\left.\begin{aligned}
x_{n+1} &= \frac{ax_n}{1+c^{(11)}x_n+c^{(12)}y_n},\\
y_{n+1} &= \frac{by_n}{1+c^{(21)}x_n+c^{(22)}y_n},
\end{aligned}\right\} \quad (n=0,1,2,\ldots). \tag{2.9}
$$

Note that the system (2.9) has four equilibrium points (see [6], [16]):

$$
E_0=(0,0)\,,\ E_x=\left(\frac{a-1}{c^{(11)}},0\right),\ E_y=\left(0,\frac{b-1}{c^{(22)}}\right),
$$

and

$$
E_+=\left(\frac{(a-1)\,c^{(22)}-(b-1)\,c^{(12)}}{c^{(11)}c^{(22)}-c^{(21)}c^{(12)}},\frac{(b-1)\,c^{(11)}-(a-1)\,c^{(21)}}{c^{(11)}c^{(22)}-c^{(21)}c^{(12)}}\right).
$$

Based on the results in [16], Theorem 4.4 in [9], Remark 2, and using Theorem 2.1, we obtain the following result on the stability of the model (2.8).

Corollary 2.1. *For the non-autonomous Lesli-Gower model (2.8) the following statements are true:*

(i) *If* $0<a<1$ *and* $0<b<1$*, then all solutions of the system (2.8) converge to* E_0*, for all points* (x_0,y_0) *in the interior of* $\mathbb{R}_+^2$*; more precisely,* E_0 *is globally asymptotically stable in* $\mathbb{R}_+^2$*.*

(ii) *If* $c^{(12)}-c^{(22)}>0$ *and* $c^{(21)}-c^{(11)}>0$*, then all solutions of the system (2.8) converge to* E_x*, for all points* (x_0,y_0) *in the interior of* $\mathbb{R}_+^2$*.*

(iii) *If* $c^{(12)}-c^{(22)}<0$ *and* $c^{(21)}-c^{(11)}<0$*, then all solutions of the system (2.8) converge to* E_+*, for all points* (x_0,y_0) *in the interior of* $\mathbb{R}_+^2$*.*

(iv) *If* $c^{(12)}-c^{(22)}>0$ *and* $c^{(21)}-c^{(11)}<0$*, then all solutions of the system (2.8) converge to* E_y*, for all points* (x_0,y_0) *in the interior of* $\mathbb{R}_+^2$*.*

In [1], the following competitive system of difference equations

$$
x_{n+1}=\frac{x_n}{a+y_n^2},\quad y_{n+1}=\frac{y_n}{b+x_n^2},\quad n=0,1,\ldots, \tag{2.10}
$$

was considered, where the parameters a and b are positive numbers, and initial conditions x_0 and y_0 are arbitrary non-negative numbers. Using linearized theory and sequence theory, it was proved that the zero equilibrium $E_0=(0,0)$ is globally asymptotically stable. Here, we will prove it using the method of Lyapunov functions, taking that

$V : \mathbb{R}^2_+ \to \mathbb{R}$ of the form $V\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2$ of the map F associated with the system (2.10). Namely, if $x \geq 0$, $y \geq 0$, $(x,y) \neq (0,0)$, $0 < a < 1$, and $0 < b < 1$, we have that

$$\begin{aligned}
\Delta V &= V\left(F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) - V\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(x\frac{1}{a+y^2}\right)^2 + \left(y\frac{1}{b+x^2}\right)^2 - x^2 - y^2 \\
&= x^2\left(\left(\frac{1}{a+y^2}\right)^2 - 1\right) + y^2\left(\left(\frac{1}{b+x^2}\right)^2 - 1\right) \\
&\leq x^2\left(\frac{1}{a^2} - 1\right) + y^2\left(\frac{1}{b^2} - 1\right) < 0.
\end{aligned}$$

Since $V\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2 \to \infty$, as $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| \to \infty$ the equilibrium point $E_0 = (0,0)$ is globally asymptotically stable when $0 < a < 1$ and $0 < b < 1$.

If we consider the following non-autonomous system

$$x_{n+1} = \frac{x_n}{a_n + y_n^2}, \quad y_{n+1} = \frac{y_n}{b_n + x_n^2}, \quad n = 0,1,..., \tag{2.11}$$

where $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then, by using Theorem 2.1, for which the system (2.10) is a limiting system, we obtain the following result.

Corollary 2.2. *All solutions of the system (2.10) globally asymptotically converge to* $E_0 = (0,0)$ *for* $0 < a < 1$ *and* $0 < b < 1$*, and for all* $x_0 \geq 0$ *and* $y_0 \geq 0$.

Now, consider the following autonomous competitive system of difference equations:

$$\begin{aligned} x_{n+1} &= a x_n e^{-\alpha y_n}, \\ y_{n+1} &= b y_n e^{-\beta x_n}, \end{aligned} \quad (n = 0,1,2,...). \tag{2.12}$$

The equilibrium points $(\overline{x}, \overline{y})$ of the system (2.12) satisfy the following system of algebraic equations:

$$\begin{aligned} \overline{x} &= a\overline{x}e^{-\alpha\overline{y}}, \\ \overline{y} &= a\overline{y}e^{-\beta\overline{x}}. \end{aligned}$$

It is easy to see that the system (2.12) has the equilibrium $E_0 = (0,0)$ for all values of the parameters. This equilibrium point is unique if $0 < a < 1$ and $0 < b < 1$. For $a > 1$, $b > 1$, $\alpha > 0$ and $\beta > 0$ the system (2.12) has a positive equilibrium $E_+ = \left(\frac{\ln b}{\beta}, \frac{\ln a}{\alpha}\right)$. If $a = 1$, then there exist infinitely many equilibrium points $E_{\overline{x}} = (\overline{x}, 0)$, $\overline{x} \geq 0$, but if $b = 1$, then there exist infinitely many equilibrium points $E_{\overline{y}} = (\overline{y}, 0)$, $\overline{y} \geq 0$.

The map associated with the system (2.12) has the following form:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} axe^{-\alpha y} \\ bye^{-\beta x} \end{pmatrix}. \tag{2.13}$$

Based on the Jacobian matrix associated with the map (2.13),

$$\left(\begin{array}{cc} ae^{-\alpha y} & -a\alpha xe^{-\alpha y} \\ -b\beta ye^{-\beta x} & be^{-\beta x} \end{array}\right),$$

we obtained the following result about the local stability of the equilibrium point E_0.

Lemma 2.2. *The following statements hold for the equilibrium point* E_0*:*

(a) *If* $0 < a < 1$ *and* $0 < b < 1$*, then* E_0 *is globally asymptotically stable.*
(b) *If* $a = 1$ *or* $b = 1$*, then* E_0 *is a non-hyperbolic.*
(c) *If* $a > 1$ *or* $b > 1$*, then* E_0 *is unstable (a saddle point or a repeller).*

Proof. The Jacobian of the map T at the equilibrium $E_0 = (0,0)$ is of the following form

$$J_T(0,0) = \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right).$$

The eigenvalues of the Jacobian at the equilibrium $E_0 = (0,0)$ are $\lambda_1 = a$ and $\lambda_2 = b$, which implies that $E_0 = (0,0)$ is locally asymptotically stable for $0 < a < 1$ and $0 < b < 1$, but is unstable (a saddle point or a repeller) if $a > 1$ or $b > 1$ and a non-hyperbolic point for $a = 1$ or $b = 1$.

If $0 < a < 1$ and $0 < b < 1$, then the first equation of the system (2.13) implies that $x_{n+1} < ax_n < a^{n+1}x_0$, which means that $x_n \longrightarrow 0$ as $n \longrightarrow \infty$ (since $x_n \geq 0$ for all $n = 0,1,\ldots$) . From the second equation of the system (2.13), we have that $y_{n+1} < bx_n$, which implies that $y_n \longrightarrow 0$ as $n \longrightarrow +\infty$ (since $y_n \geq 0$ for all $n = 0,1,\ldots$), that is, $E_0 = (0,0)$ is a global attractor. Since $E_0 = (0,0)$ is locally asymptotically stable, we conclude it is globally asymptotically stable. □

Remark 2.2. By using the Jacobian matrix associated with the map (2.13), we have that the following statements are true:

1. If $a = 1$, then every equilibrium point $E_{\overline{x}}$, $\overline{x} \geq 0$ is non-hyperbolic.
2. If $b = 1$, then every equilibrium point $E_{\overline{y}}$, $\overline{y} \geq 0$ is non-hyperbolic.
3. If $a > 1$ and $b > 1$, then E_+ is unstable (a saddle point or a repeller).

Note that the system (2.13) is a limiting system of the following non-autonomous competitive system:

$$\begin{array}{l} x_{n+1} = a_n x_n e^{-\alpha_n y_n}, \\ y_{n+1} = b_n y_n e^{-\beta_n x_n}, \end{array} \quad (n = 0,1,2,\ldots), \tag{2.14}$$

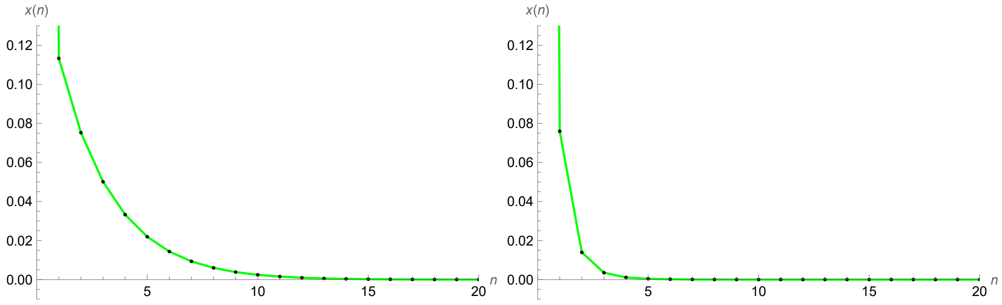
where $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, $\lim\limits_{n\to\infty} \alpha_n = \alpha$ and $\lim\limits_{n\to\infty} \beta_n = \beta$.

We obtain the following result using Lemma 2.2 and Theorem 2.1.

Corollary 2.3. *If* $0 < a < 1$, $0 < b < 1$, $\alpha > 0$ *and* $\beta > 0$*, then all solutions of the system (2.14) globally asymptotically converge to* $E_0 = (0,0)$ *for all* $x_0 \geq 0$ *and* $y_0 \geq 0$*.*

Example 2.1. *It seems interesting to compare the behavior of the autonomous system solutions (2.10) for* $a = 0.9$ *and* $b = 0.99$*:*

$$x_{n+1} = \frac{x_n}{0.9 + y_n^2}, \quad y_{n+1} = \frac{y_n}{0.99 + x_n^2}, \quad n = 0,1,\ldots, \tag{2.15}$$



(A) (B)

FIGURE 1. Time series of the components x_n of the systems in Example 2.1: (A) autonomous case; (B) non-autonomous case (with initial values $x_0 = 2.1, y_0 = 4.2$).

with the solutions of the corresponding non-autonomous system (2.11) with the coefficients $a_n = \frac{0.9n+10}{n+1}$ *and* $b_n = 0.99 + \frac{1}{n}$*, that is:*

$$x_{n+1} = \frac{x_n}{\frac{0.9n+10}{n+1} + y_n^2}, \quad y_{n+1} = \frac{y_n}{0.99 + \frac{1}{n} + x_n^2}, \quad n = 0, 1, \dots . \tag{2.16}$$

What is unexpected in this case is the faster convergence of the solution of the non-autonomous system compared to the autonomous system, especially of the components x_n*. In both cases, the components* y_n *converge to* 0 *quickly (Figure 1).*

3. CONCLUSION

Relying on previous research, where theorems of global attractiveness of some classes of non-autonomous competitive systems of difference equations with asymptotically constant coefficients were proved, this paper presents a general theorem for an arbitrary non-autonomous competitive system. The obtained results were applied to three typical cases. In the end, the rate of convergence of the solution of a non-autonomous competitive system of difference equations was compared with the convergence of the solution of the corresponding limiting autonomous system. In doing so, the unexpected conclusion was reached that the solutions of a non-autonomous competitive system can converge to the equilibrium point even faster than the solutions of its limiting autonomous system.

REFERENCES

[1] Dž. Burgić, M.R.S. Kulenović and M. Nurkanović, Global Dynamics of a Rational System of Difference Equations in the Plane, *Communications on Applied Nonlinear Analysis*, **15** (1) (2008), 71-84.

[2] M. Garić-Demirović, M.R.S. Kulenović and M.Nurkanović, Global Behavior of Four Competitive Rational Systems of Difference Equations in the Plane, *Discrete Dynamics in Nature and Society*, Volume 2009, Article ID 153058, 34 pages.

[3] M. Garić-Demirović, M.R.S. Kulenović and M. Nurkanović, Global Behavior of Two Competitive Rational Systems of Difference Equations in the Plane, *Communications on Applied Nonlinear Analysis*, 16 (2009), No. 3, 1-18.

[4] M.R.S. Kulenović and E. Bertrand, Global Dynamic Scenarios for Competitive Maps in the Plane, *Advances in Difference Equations*, (2018), 2018: 28p.

[5] M.R.S. Kulenović, J. Marcotte, and O. Merino, Properties of Basins of Attraction for Planar Discrete Cooperative Maps, *Discrete Contin. Dyn. Syst. B*, 26 (2021), 2721–2737.

[6] M.R.S. Kulenović and D.T. McArdle, Global Dynamics of Leslie-Gower Competitive Systems in the Plane, *Mathematics,* **7** (1), 76 (2019). https://doi.org/10.3390/math7010076

[7] M.R.S. Kulenović and O. Merino, Invariant Curves for Planar Competitive and Cooperative Maps, *Journal of Difference Equations and Applications*, 24 (2018) 898-915.

[8] M.R.S. Kulenović, O. Merino, and M. Nurkanović, Global Dynamics of Certain Competitive Systems in the Plane, *Journal of Difference Equations and Applications*, Vol. 18, No. 12 (2012), 1951-1966.

[9] M.R.S. Kulenović, M. Nurkanović, Z. Nurkanović, and S. Trolle, Asymptotic Behavior of Certain Non-autonomous Planar Competitive Systems of Difference Equations, *Mathematics* (MDPI), 11, 3909 (2023), 22 p. https://doi.org/10.3390/math11183909

[10] M.R.S. Kulenović, M. Nurkanović, Z. Nurkanović, and S. Trolle, Stability of certain non-autonomous cooperative systems of difference equations with the application to evolutionary dynamics, (2024) (will appear).

[11] M.R.S. Kulenović and M. Nurkanović, Global asymptotical behavior of a two dimensional system of difference equations modeling cooperation, *Journal of Difference Equations and Applications*, 9 (2003), 149-159.

[12] M.R.S. Kulenović and M. Nurkanović, Asymptotical behavior of a system of linear fractional difference equations, *Journal of Inequalities and Applications*, Vol. 2005, No.2 (2005), 127-143.

[13] M.R.S. Kulenović and M. Nurkanović, Asymptotic Behavior of a Competitive System of Linear Fractional Difference Equations, *Advances in Difference Equations*, Volume 2006, Article ID 19756, Pages 1-13.

[14] M.R.S. Kulenović and M. Nurkanović, Global Behavior of a Two-dimensional Competitive System of Difference Equations with Stocking, *Mathematical and Computer Modelling,* 55 (2012), 1998-2011.

[15] M.R.S. Kulenović and S. Van Beaver, Global Dynamics of a Cooperative System with Ceiling Density Dependence, *International J. Difference Equations*, 14 (2019), 59-74.

[16] K. Mokni, S. Elaydi, M. CH-Chaui, and A. Eladdadi, Discrete evolutionary population models: a new approach, *Journal of Biological Dynamics*, **14** (1), (2020), 454-478. https://doi.org/10.1080/17513758.2020.1772997

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