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USING MATHEMATICAL SOFTWARE FOR HYPERBOLIC PARABOLOIDS IN BUILDING DESIGN

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Dedicated to the 75th birthday of our dear Professor Mirjana Vuković

ABSTRACT. In the modern world it is impossible to imagine an environment without concrete, buildings, asphalt tracks, banks, and the like. Throughout history, the environment has changed and developed due to human desire for intellectual and material progress. From flat to minimal surfaces, knowledge of surfaces is closely related to construction. Builders and planners used new forms in construction, and the ball, circle, sphere, and their parts were used as a basis. As forms were developed, so were new materials, but stone remained a building material for thousands of years. The emergence of new building materials is often combined with new forms. Until then, the usual forms were spheres, rollers, planes, and surfaces. The 20th century was characterized by both theoretical study and the use of plate surfaces in construction. The special feature of these surfaces is that they support themselves. One such surface is a hyperbolic paraboloid, and the reason for this is simple. The hyperbolic paraboloid is a surface of great application, great possibilities, and enviable aesthetic value. Its application is wide; however, it is mostly used as a roof surface. The application of mathematics as a science in various fields with the help of programming mathematical tools and the like is becoming increasingly common. This paper was created as a result of multiple applications of mathematics as a science, where two software packages, MATLAB and Wolfram Mathematica, were used. A hyperbolic paraboloid in space, mutual combinations of two or more equal paraboloids with different dimensions, and their application in construction are presented. In this work, a hyperbolic paraboloid is presented in a new form, a form that carries a new age of construction. The work is divided into three parts: the mathematical, the software, and the structural part.

1. INTRODUCTION

In our rapidly evolving world, using mathematics as a fundamental science is increasingly pervasive across diverse fields. Accompanied by the aid of mathematical programming tools and similar technologies its application extends into realms such as engineering, architecture, and construction. Central to our discourse is exploring plate surfaces and their geometric constructs within the context of construction endeavors.

Among the repertoire of plate surfaces, the hyperbolic paraboloid emerges as a prominent representative of second-order surfaces. Its inherent properties and geometric characteristics render it a compelling subject for further investigation. In this exploration,

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we delve into the conceptualization and application of the hyperbolic paraboloid in spatial contexts, examining both singular surfaces and their amalgamations in construction practice.

This inquiry unveils the simplicity of constructing hyperbolic paraboloids juxtaposed against their broad utility and aesthetic appeal. As we showcase these surfaces in novel configurations, we illuminate their role as harbingers of a new era in architectural and construction paradigms. Through meticulous examination and creative application, we embark on a journey to elucidate the transformative potential of the hyperbolic paraboloid within the fabric of contemporary building construction.

2. THE HYPERBOLIC PARABOLOID AND ITS REAL-WORLD APPLICATION

Hyperbolic paraboloids are a canonical example of a surface with a ”saddle point”, i.e., a stationary point that is neither a maximum nor a minimum. At such points on the surfaces the Gaussian curvature is negative. The name ”hyperboloid” derives from the fact that the vertical sections of this surface are parabolas, while the horizontal cross sections are hyperbolas. However, even the vertical sections are more complex than the elliptical paraboloid. The general form of the hyperboloid equation is given by:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}. \tag{2.1}$$

Now let’s explore a couple of real-world applications of the hyperbolic paraboloid. In construction, hyperbolic buildings use less material compared to other conical shapes. They offer greater stability against external forces than flat buildings. Despite their decorative effect, hyperbolic structures often exhibit low space efficiency [1].

For instance, hyperbolic structures find widespread use in the cooling towers of power plants and industrial facilities. Their shape facilitates efficient air circulation and heat dissipation. The upward draft created by the hyperboloid’s conical shape enables effective cooling of water or gases, making it an indispensable component in thermal power plants and industrial processes. Additionally, the hyperbolic shape enhances airflow through the cooling tower. It contributes to the strength and stability of tall structures, as cooling towers must release steam into the atmosphere from a significant height. The Kobe Port Tower boasts an hourglass shape featuring two hyperbolas. This symmetry ensures that views from one side mirror those from the opposite side (Figure 1).

FIGURE 1. The Kobe Port Tower

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In antenna systems, the hyperboloid shape offers advantages for telecommunications and radar applications (Figure 2).

FIGURE 2. The antenna system

It provides a wide radiation pattern, improving signal coverage. Hyperboloid reflectors and arrays are utilized in radio astronomy, satellite communications, and wireless networks for efficient signal transmission and reception over long distances. Regarding lamp design, bed lights typically have a cylindrical shape. However, when illuminated, they cast a unique, often hyperbolic shade on the wall behind them. This effect occurs because these lights usually open at both the top and bottom, resulting in circular light scattering intersected by an ordinary wall, creating a hyperbolic shade. Such forms are frequently employed for wall decoration (Figure 3).

FIGURE 3. Lamp

3. APPLICATION OF THE HYPERBOLIC PARABOLOID WITH THE HELP OF MATHEMATICAL SOFTWARE

This paper was created as a result of various applications of mathematics as a science. In this paper we used two software packages, MATLAB and Wolfram Mathematica, which is why we will talk about them briefly.

3.1. Wolfram Mathematica

The hyperbolic paraboloid is a surface characterized by its saddle shape, where the curvature is negative along one axis and positive along the other. In this significance test we utilize Wolfram Mathematica to analyze critical points on the surface and determine their nature-whether they represent local minima, local maxima, or saddle points. First we define the equation of the hyperbolic paraboloid and visualize it using Mathematica's plotting capabilities. Then we compute the partial derivatives of the hyperbolic paraboloid concerning its variables, x and y. Next we identify critical points by solving for the points where both partial derivatives are equal to zero. These critical points represent potential extrema or saddle points on the surface. To determine the nature of each critical point we compute the Hessian matrix-a square matrix of second-order partial derivatives-at each critical point. The eigenvalues of the Hessian matrix provide valuable information about the curvature of the surface at each critical point. If both eigenvalues are positive, the critical point represents a local minimum. Conversely, if both eigenvalues are negative, the critical point is a local maximum. If the eigenvalues have opposite signs, the critical point is a saddle point. By performing this significance test using Wolfram Mathematica we gain valuable insights into the geometric properties of the hyperbolic paraboloid and its critical points, facilitating further analysis and understanding of this important mathematical surface. One example of a hyperbolic paraboloid in the Wolfram Mathematica software, where the function is given in the form $z = x^2 - y^2$, ranging from -1 to 0.1, is provided in parametric form (Figure 4). The details can be found in [3], [4] and [5].

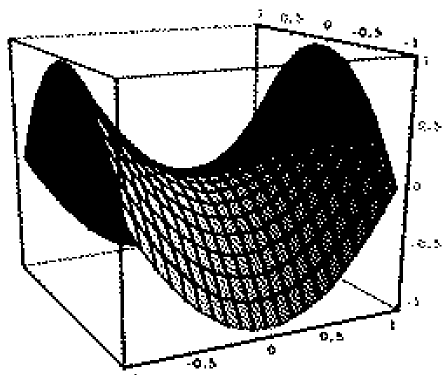

FIGURE 4. Hyperbolic paraboloid

3.2. **MATLAB**

As a computational tool widely used in engineering and scientific research, MATLAB plays a significant role in our analysis and manipulation of hyperbolic paraboloids. MATLAB provides us with a comprehensive environment for numerical analysis, allowing us to perform calculations and simulations related to hyperbolic paraboloids with precision and efficiency. Its extensive library of built-in functions and toolboxes enables us to solve complex mathematical problems associated with hyperbolic paraboloids. In addition to numerical analysis, MATLAB offers us robust visualization capabilities, making it easier to visualize and interpret the geometric properties of hyperbolic paraboloids. We can generate 3D plots, contour plots, and surface plots to gain insights into the shape, curvature and behavior of hyperbolic paraboloids (see Figure 5, [5]).

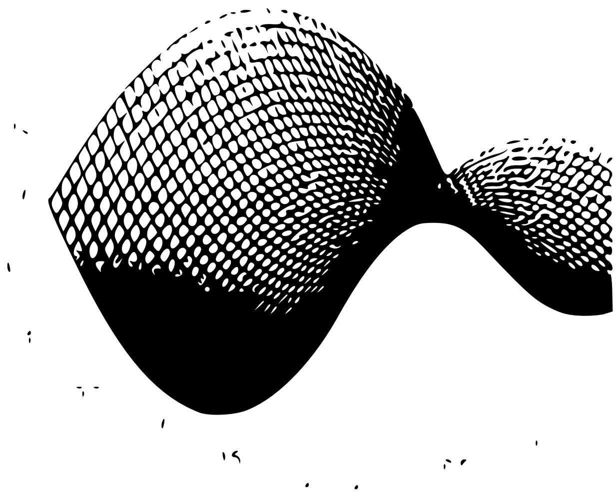

FIGURE 5. Hyperbolic paraboloid

MATLAB allows us to perform parametric modeling of hyperbolic paraboloids, enabling us to define and manipulate the parameters that govern the shape and characteristics of the surface. This flexibility facilitates our exploration of various configurations and designs of hyperbolic paraboloids for different applications. Furthermore, MATLAB includes optimization algorithms that we can apply to optimize parameters such as dimensions, curvature and orientation of hyperbolic paraboloids for specific objectives or constraints. This capability is particularly valuable in engineering and design tasks where maximizing performance or minimizing costs is essential. MATLAB seamlessly integrates with other software tools and programming languages, allowing us to combine the capabilities of MATLAB with those of other software packages for comprehensive analysis and design of hyperbolic paraboloids. This interoperability enhances our productivity and facilitates interdisciplinary collaboration.

4. APPLICATION OF THE HYPERBOLIC PARABOLOID TO THE CONSTRUCTION OF THE ROOF SYSTEM OF RESIDENTIAL BUILDINGS

To have a clear process of how to apply a hyperbolic paraboloid to the roof of an object it is necessary to first draw a 3D situation and a project of the roof (Figure 6).

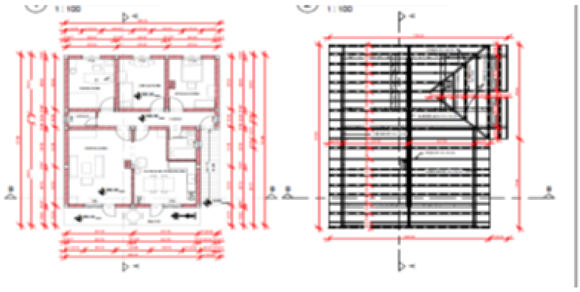

FIGURE 6. 3D situation of the roof of the buildings [6]

In the context of a roof structure, the equation can be slightly modified to suit the orientation and dimensions of the roof. Let's say the roof is aligned with the x and y axes, and the highest point (top) of the roof is at the start line (0,0,0). Then the equation of the hyperbolic paraboloid becomes

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2} + h, \tag{4.1}$$

h is the height of the peak above the $x - y$ plane (see Figure 7, [2]).

FIGURE 7. Hyperbolic paraboloid of the roof structure of a residential building [6]

This equation describes the shape of the roof surface. To create the physical structure, we should define the dimensions of the roof (length, width, and height) and then use this equation to generate the appropriate curvature for the roof surface. In an architectural and engineering context, hyperbolic paraboloid roofs are often constructed using flat beams or cables arranged in a transverse pattern to form a hyperbolic paraboloid shape. The mathematical equations that determine the geometry of these beams or cables would be more complex and would involve concepts from structural engineering and statics. The core is precisely reflected in the mathematical application of geometry to the object. By observing the project of a building it was noticed that the roof construction is very unstable and over time the roof would have to be changed in the next 3-5 years due to various everyday influences. For example, if you hold a sheet of paper in your hand, it bends and cannot support its weight. That same sheet of paper, if squeezed or slightly curved upwards, becomes able to support its weight. The upward curvature increases the stiffness and load-bearing capacity, thereby moving part of the material away from the neutral axis. The same can be applied to the roof of the building. The load-bearing capacity of the roof structure depends on the curvature and specific curvature. When a hyperbolic paraboloid is tilted in the direction of its generating lines, it essentially means that it is rotated or tilted along one or both of its axes. Let's consider the case when it is tilted along one axis, let's say the x-axis. We can achieve this slope by introducing a rotation matrix into the equation, using the general equation of the hyperbolic paraboloid (Figure 8).

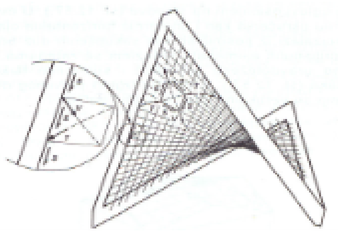

FIGURE 8. A hyperbolic paraboloid leaning in the direction of the directions that generate it [6]

A roof composed of hyperbolic paraboloids (hypers) usually includes multiple hyperunits arranged together to form an overall structure. Each hyperunit itself can be described mathematically using the equation of a hyperbolic paraboloid. The parameters would have to be determined based on the specific design requirements and constraints of the roof structure. In order to create a continuous roof surface additional considerations such as how the hyperunits are connected and joined together would also have to be solved mathematically, often through computational techniques, geometry and architectural design (Figure 9).

FIGURE 9. A view of the roof composed of hypers [6]

A roof composed of conoidal hyperbolic paraboloids refers to a structure in which multiple hyperbolic paraboloid units are arranged to form a conical shape. Each unit of the hyperbolic paraboloid can be described mathematically by the equation of the hyperbolic paraboloid. However, to represent the conoid shape we need to introduce additional parameters to control the position, orientation, and scale of each hyperbolic paraboloid unit (Figure 10).

FIGURE 10. Colloidal hyperbolic paraboloid roof [6]

5. IMPLEMENTATION OF HYPERBOLIC PARABOLOID ON ROOF SYSTEMS USING MATHEMATICAL SOFTWARE

Now based on the above consideration, in our paper we can create animations that will solve the observed problems. For the sake of simplicity, animations coded in mathematical software are shown in the paper in the form of figures.
An animation made in MATLAB showing the construction of a hyperbar (hyperbolic paraboloid) in three directions proceeds as follows. The animation begins with the initialization of the plot window in MATLAB. All the necessary parameters and variables are defined, including the dimensions of the hyper, the number of steps for each direction and all other parameters relevant to the construction. This grid consists of points in 3D space that will define the shape of the hyper. These points are calculated based on the equations that describe the hyper in three directions. The animation then proceeds to gradually construct the hyper surface. This can be done by iteratively adjusting the parameters of the hyper equations to create a smooth transition from the flat mesh to the final hyper shape. Each iteration updates the position of the grid points according to the evolving hyper equations. As the hyper is constructed, it is visualized in the MATLAB plot window. The graph is updated at each iteration to show the current state of the hyper surface. The visualization could include wireframe rendering or surface rendering to show the shape of the hyper more clearly. When the hyper construction is complete the animation ends showing the fully constructed surface of the hyper. At this stage any finishing touches or visualization adjustments can be made to ensure the clarity and accuracy of the final result (Figure 11).

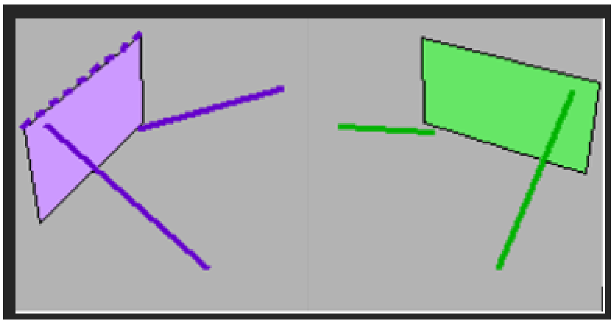

FIGURE 11. Hyper construction using three directions

To begin constructing the hyperbolic paraboloid (hyper) using MATLAB, we would initialize the MATLAB environment and set up the graphics window. This entails defining parameters such as hyperdimensions, the number of steps for each plane and any other relevant variables. Next we would generate three planes in 3D space to intersect at right angles, forming a framework for constructing the hyper. Each plane would be represented by a set of points or vertices. The points of intersection of these three planes would then be calculated, as they lie on the surface of the hyper and are crucial for its construction. Subsequently, a grid or matrix of points on the surface of the hyper would be created based on the intersection points of the three planes, effectively defining the shape of the hyper. Through iterative processes, each point on the grid would be processed to calculate its position using the equations of the hyperbolic paraboloid. This calculation determines the height (z-coordinate) of each point based on its x and y coordinates. Throughout this process, the MATLAB graph would be continuously updated to visualize the evolving shape of the hypersurface. Various rendering techniques such as wireframe or surface rendering can be employed to enhance clarity. By leveraging MATLAB's animation functions the plot would be updated at each iteration to depict the current state of the hyper-construction. This would create a smooth transition between frames, effectively illustrating how the hyper gradually takes shape over time (Figure 12).

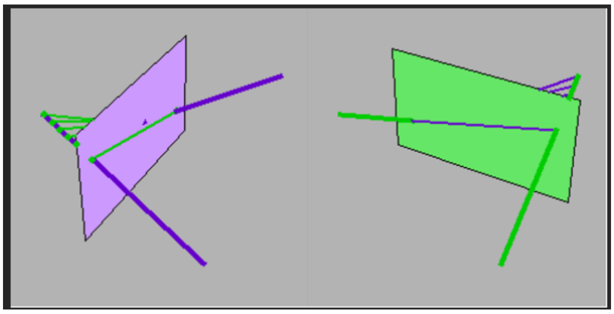

FIGURE 12. Construction using three planes

MATLAB Code for Animating the Hyperbolic Paraboloid:

Define the range and parameters for the plot

$$x = linspace(-5, 5, 100);$$

$$y = linspace(-5, 5, 100);$$

$$[X, Y] = meshgrid(x, y);$$

Hyperbolic paraboloid equation

$$Z = X.^2 - Y.^2;$$

Create a figure for the animation

$$figure;$$

$$axis([-55-55-2525]);$$

$$hold\ \ on;$$

Plot the hyperbolic paraboloid surface

$$hSurface = surf(X, Y, Z);$$

$$shading\ \ interp;$$

$$colormap(jet);$$

Define animation parameters

$$numFrames = 100;$$

$$angleStep = 360/numFrames;$$

Example construction of rulings (generatrices) for two systems

System 1 rulings

$$for\ \ t = -5:0.5:5$$
$$lineX = linspace(-5,5,100);$$
$$lineY = t * ones(1,100);$$
$$lineZ = lineX.^2 - lineY.^2;$$
$$plot3(lineX, lineY, lineZ, 'k', 'LineWidth', 1);$$
$$end$$

System 2 rulings

$$t = -5:0.5:5$$
$$lineX = t * ones(1,100);$$
$$lineY = linspace(-5,5,100);$$
$$lineZ = lineX.^2 - lineY.^2;$$
$$plot3(lineX, lineY, lineZ, 'r', 'LineWidth', 1);$$
$$end$$

Animation loop

$$for\ \ k = 1:numFrames$$

Rotate the view

$$view(angleStep * k, 30);$$

Update the plot

$$drawnow;$$

Pause for a short duration to control the speed of the animation

$$pause(0.05);$$
$$end$$

Optionally, save the animation as a video

$$v = VideoWriter('hyperbolic_paraboloid.avi');$$
$$open(v);$$
$$for\ \ k = 1:numFrames$$
$$view(angleStep * k, 30);$$
$$frame = getframe(gcf);$$
$$writeVideo(v, frame);$$
$$drawnow;$$
$$end$$
$$close(v);$$

In the process of animating a hyperbolic paraboloid (hyper) construct using winged four-pointers in MATLAB, our team begins by setting up the environment, initializing parameters and defining variables relevant to the animation (Figure 13). These parameters include the dimensions of the hyper, the number of steps for the animation stages and other important variables. We then proceed to define the geometry of the four-pointer wing structure, typically consisting of four intersecting planes arranged in a specific configuration to create a hyper shape. Each plane is represented by its equation or set of points in 3D space. Subsequently, we calculate the intersection points of the four planes. These points serve as the vertices of the hyper, determining its overall shape and structure. Using these intersection points, we generate a grid or matrix of points representing the hyper surface. These points are crucial for defining the surface of the hyper and form the basis for its construction. Throughout the animation process, we progress through each stage of the construction, gradually adjusting the parameters or positions of the intersection points to ensure a smooth transition from the initial state to the final hyper shape. At each iteration, the MATLAB plot is updated to visualize the ongoing construction of the hyper. This visualization may employ techniques such as wireframe or surface rendering to effectively display the evolving hyperstructure. To achieve fluid animation, the plot is updated at regular intervals, illustrating the progressive development of the hyper construction over time. MATLAB's animation functions provide control over frame timing and appearance, ensuring coherent and visually appealing animation. Once the construction of the hyper is complete, we finalize the animation, showcasing the fully constructed surface of the hyper.

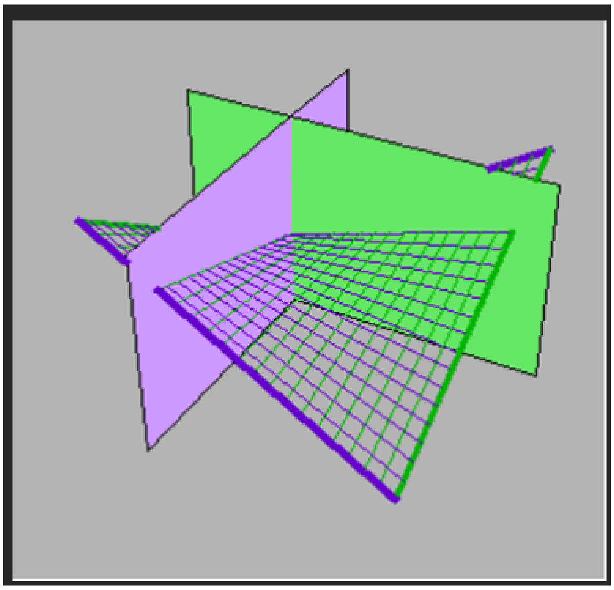

FIGURE 13. Construction using four tops

In the context of MATLAB animation, the construction of a hyperbolic paraboloid (hyper) using translation surfaces involves several key steps (Figure 14):

1. It starts by setting up the MATLAB environment and initializing parameters such as hyper dimensions, number of animation steps and any other relevant variables.
2. Create translation surfaces that serve as a framework for constructing the hyper. These surfaces can be generated by translating a curve or profile along two

orthogonal directions in 3D space. These translation surfaces will intersect to form a hypershape.
3. Calculate the intersection points of the translational surfaces. These points will define the peaks of the hipper and determine its overall shape.
4. Generate a grid or grid of points on the hypersurface. These points will be used to construct the hyper and provide the basis for its visualization.
5. Iterate through each step of the construction process, adjusting the parameters or positions of the intersection points to gradually build the hyper shape. This involves translating and transforming translation surfaces to create a hyperbolic paraboloid.
6. Update the MATLAB plot at each iteration to visualize the ongoing hyper-construction. Animation functions in MATLAB are used to create smooth transitions between frames, illustrating the progressive development of the hyper structure.
7. When the hyper structure is complete, integrate it into the roof structure by adding support elements such as beams or columns.

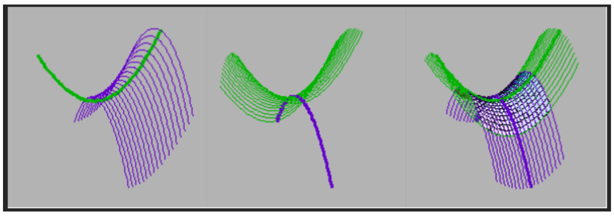

FIGURE 14. Construction using translation surfaces

All animations featured in this paper can be accessed via the following
https://drive.google.com/drive/folders/1qhKCIr_Nz7qPNUbSuXrMMwGbV8n-57M3?usp=drive_link

6. CONCLUSION

Hyperbolic paraboloids offer a versatile and aesthetically pleasing solution for roof construction capable of covering large areas with minimal support structures due to their inherent structural stability. Their unique geometric shape not only adds architectural interest but also enhances the visual appeal of buildings, contributing to sophisticated design in both modern and historic architecture. Hyperbolic paraboloids efficiently distribute the load and enable economical construction while maintaining structural integrity, making them particularly suitable for large-span roof structures in residential and commercial buildings. Additionally, their geometric shape facilitates the integration of skylights and terrace windows, allowing sufficient natural light and ventilation, which can improve people's comfort and energy efficiency. Hyper roofs can be designed to accommodate green roof systems, solar panels, and rainwater harvesting systems, promoting sustainability and environmental responsibility in building design. Looking ahead, further research could explore advanced computational methods to optimize hyperdesign, innovative construction materials, and integration of hyper with new technologies such as parametric design and digital manufacturing. Studies on the long-term performance and maintenance of hyper roofs could provide valuable insight for future architectural practice and urban planning. By embracing the potential of hyperbolic paraboloids, we can continue to push the boundaries of architectural innovation and create more sustainable and aesthetically pleasing built environments for generations to come.

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